hochschule für angewandte wissenschaften
FACHBEREICH FAHRZEUGTECHNIK UND FLUGZEUGBAU hamburg
university of applied sciences

Unterlagen
zur Vorlesung

Flugmechanik 1

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Vorwort

Flugmechanik 1 an der HAW Hamburg beschäftigt sich schwerpunktmäßig mit Flugleistungen (aircraft performance) darüber hinaus wird ein Einstieg in Flugeigenschaftsrechnungen (aircraft stability and control) gegeben.

Die Vorlesung *Flugmechanik 1* wird seit SS99 durchgeführt basierend auf dem **Flugmechanik-Skript** der *University of Limerick, Department of Mechanical & Aeronautical Engineering*. Der Autor des Skripts ist **Trevor Young**. Mr. Trevor Young ist an der University of Limerick verantwortlich für die Organisation der Kurse im Flugzeugbau. Er lehrt die Fächer Flugmechanik und Flugzeugentwurf.

Die hier vorliegenden Unterlagen sind als Ergänzung gedacht zum Skript. Die Numerierung der Unterlagen entspricht der Numerierung im Skript. **Ich erstelle die Unterlagen in englisch, da sie auch international genutzt werden.**

Contents

1 Introduction to Flight Mechanics and the ISA

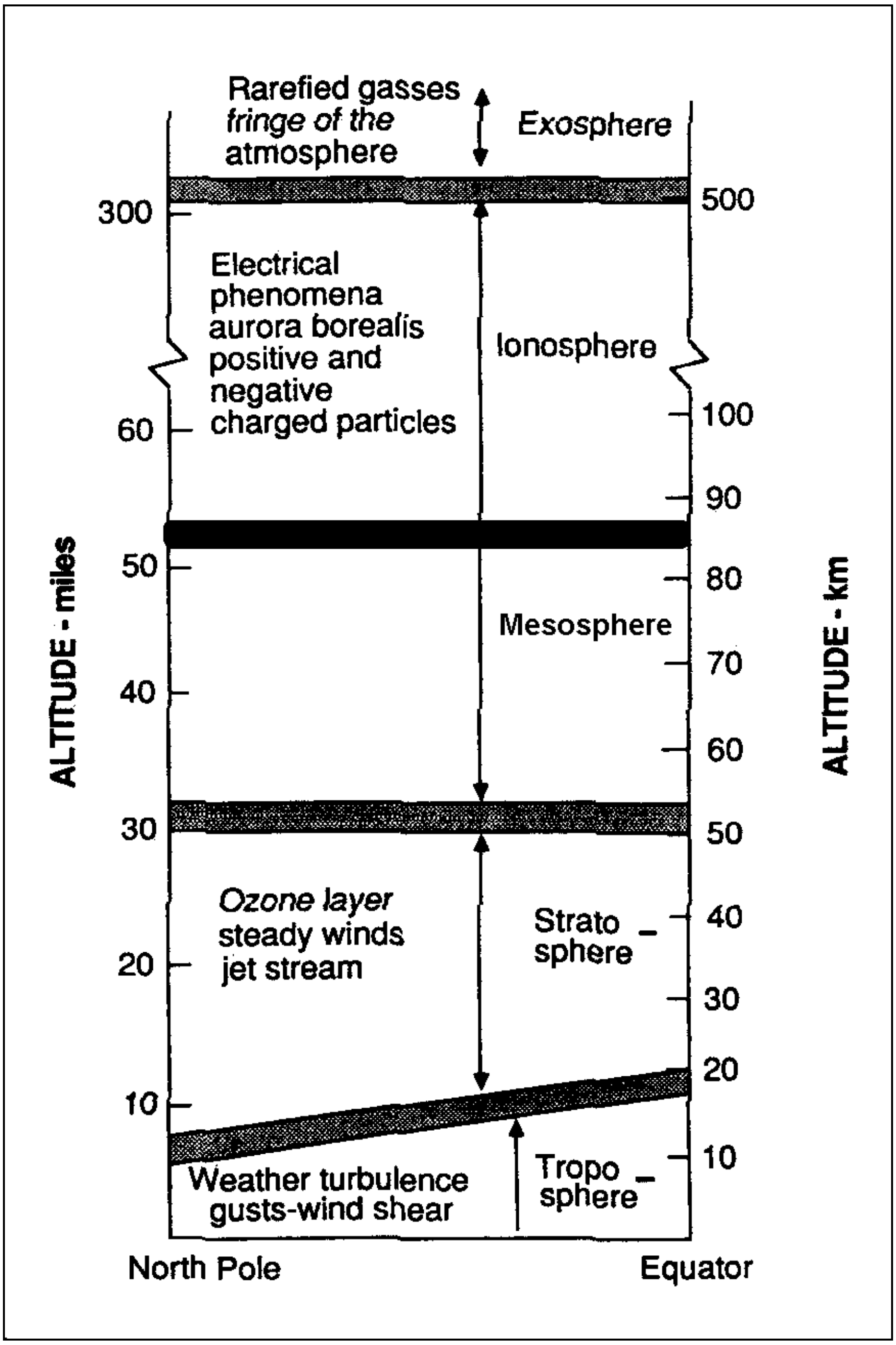


Fig. 1.1: Basic classification of the atmosphere [15][1] (modified)

[1] Numbering of references according to lecture notes. Additional references are listed in a separate Chapter in the back of these notes.

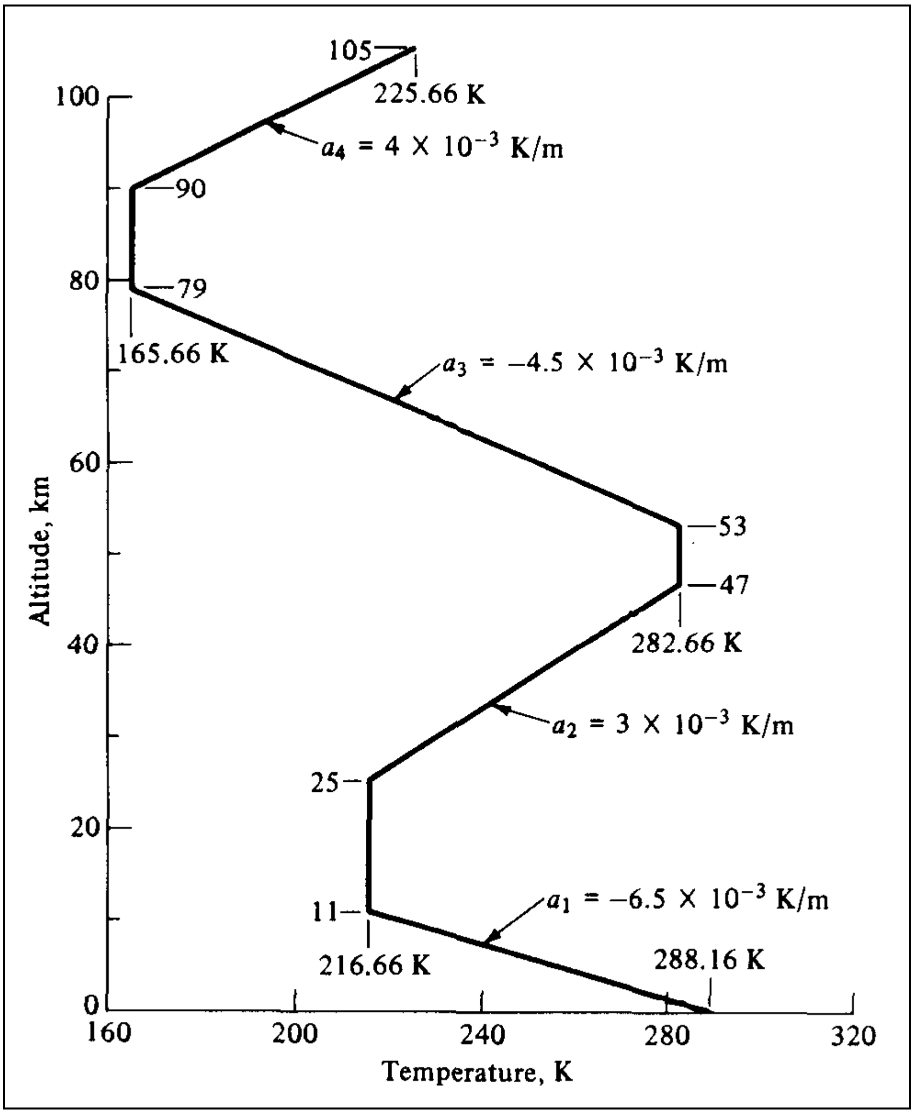


Fig. 1.2: Temperature distribution in the standard atmosphere (**ANDERSON 1989**)

1.1 Equations for the International Standard Atmosphere

Equations for the troposphere

Troposphere from 0 m = 0 ft **to** 11000 m = 36089 ft (geopotential height)

$$T = T_0 - L \cdot H$$

H		geopotential height
T_0	=	288.15 K
L	=	0.0065 K/m = 6.5 K/km = 1.9812 · 10^{-3} K/ft

$$\delta = \frac{p}{p_0} = \left(1 - k_a \cdot H\right)^{5.25588}$$

$$\sigma = \frac{\rho}{\rho_0} = \left(1 - k_a \cdot H\right)^{4.25588}$$

p_0	=	101325 Pa = 1013.25hPa = 1.01325 bar
ρ_0	=	1.225 kg/m³
k_a	=	2.2558 · 10^{-5} 1/m = 0.022558 1/km = 6.8756 · 10^{-6} 1/ft

Equations for the Stratosphere

Stratosphere from 11000 m = 36089 ft **to** 20000 m = 65617 ft (geopotential height)

$T = T_S$ = 216.65 K = -56.5 °C = const

$$\frac{\sigma}{\sigma_T} = \frac{\rho}{\rho_T} = \frac{\delta}{\delta_T} = \frac{p}{p_T} = e^{-k_b \cdot \left(H - H_T\right)}$$

H_T	=	11000 m = 11 km = 36089 ft
k_b	=	1.57688 · 10^{-4} 1/m = 0.157688 1/km = 4.80634 · 10^{-5} 1/ft
σ_T	=	0.297070
ρ_T	=	0.3639 kg/m³
δ_T	=	0.223356
p_T	=	22632 Pa = 226.32 hPa = 0.22632 bar

Equations valid for both Troposphere and Stratosphere

speed of sound $a = \sqrt{\gamma \cdot R} \cdot \sqrt{T}$ $\qquad \sqrt{\gamma \cdot R} = 20.0468 \ 1/\sqrt{\mathrm{K}} \cdot \frac{\mathrm{m}}{\mathrm{s}}$

$R = 287.053 \ \frac{\mathrm{J}}{\mathrm{kg} \cdot \mathrm{K}}$ $\qquad \gamma = 1.4$

$a_0 = 340.294 \text{ m/s} = 1225.06 \text{ km/h} = 661.48 \text{ kt}$

dynamic viscosity $\mu = \frac{\beta_S \cdot T^{3/2}}{T + S}$ $\qquad \beta_S = 1.458 \cdot 10^{-6} \ \frac{\mathrm{kg}}{\mathrm{m} \cdot \mathrm{s} \cdot \sqrt{\mathrm{K}}}$

$\mu_0 = 1.7894 \cdot 10^{-5} \ \frac{\mathrm{kg}}{\mathrm{m} \cdot \mathrm{s}}$ $\qquad S = 110.4 \text{ K}$

kinematic viscosity $\nu = \frac{\mu}{\rho}$ $\qquad \nu_0 = 1.4607 \cdot 10^{-5} \ \frac{\mathrm{m}^2}{\mathrm{s}}$

relative density $\sigma = \frac{\rho}{\rho_0}$

relative pressure $\delta = \frac{p}{p_0}$

relative temperature $\theta = \frac{T}{T_0}$

equation of state for a perfect gas $\frac{p}{\rho} = R \cdot T$ $\qquad R = 287.053 \ \frac{\mathrm{J}}{\mathrm{kg} \cdot \mathrm{K}}$

$$\frac{\delta}{\sigma} = \theta$$

1.2 Height Scales and Conversions

geometric height	h	as measured with a tape
geopotential height	H	equivalent potential energy in a uniform gravitational field with $g = 9.80665 \text{ m/s}^2$
pressure height	h_p	equivalent height for given pressure in the ISA
density height	h_ρ	equivalent height for given density in the ISA

altimeter setting		altimeter reading
1013 hPa	standard pressure	altitude above 1013 hPa pressure plain
QNH	actual pressure at mean sea level	altitude above mean sea level
QFE	actual pressure at the airfield	altitude above airfield

Definition: **F**light **L**evel = altitude above 1013 hPa pressure plain in ft / 100 ft
Example: FL 100 = 10000 ft / 100 ft

Conversion among different height definitions, pressure, density and temperature in the troposphere

STEP 1: We assume p_0 = 1013.25 hPa (standard condition) but $T = T_{ISA} + \Delta T$.

$$r_{earth} = 6.371 \cdot 10^6 \text{ m} = 6.371 \cdot 10^3 \text{ km} = 2.090 \cdot 10^7 \text{ ft}$$

$$g = g_0 \cdot \left(\frac{r_{earth}}{r_{earth} + h} \right)^2 \qquad g_0 = 9.80665 \frac{\text{m}}{\text{s}^2}$$

p

$$p = p_0 \left(1 - \frac{L}{T_0} \cdot h_p \right)^{5.25588} \qquad h_p = \frac{T_0}{L} \cdot \left(1 - \left(\frac{p}{p_0} \right)^{\frac{1}{5.25588}} \right)$$

h_p

$$\frac{h_p}{H} = \frac{T_0}{T_0 + \Delta T}$$

T —

$$T = T_0 - L \cdot H + \Delta T \qquad H = \frac{T_0 - T + \Delta T}{L}$$

— H —

$$H = \frac{r_{earth} \cdot h}{r_{earth} + h} \qquad h = \frac{r_{earth} \cdot H}{r_{earth} - H}$$

— h

$$p = p_0 \left(1 - \frac{L}{T_0 + \Delta T} \cdot H \right)^{5.25588} \qquad H = \frac{T_0 + \Delta T}{L} \cdot \left(1 - \left(\frac{p}{p_0} \right)^{\frac{1}{5.25588}} \right)$$

p

$$\frac{p}{\rho} = R \cdot T$$

ρ

$$\rho = \rho_0 \left(1 - \frac{L}{T_0} \cdot h_\rho \right)^{4.25588} \qquad h_\rho = \frac{T_0}{L} \cdot \left(1 - \left(\frac{\rho}{\rho_0} \right)^{\frac{1}{4.25588}} \right)$$

h_ρ

Fig. 1.3: Conversion among different height definitions, pressure, density and temperature in the troposphere

STEP 2: If $p_0 \neq 1013.25$ hPa this can be accounted for as indicated in **Fig. 1.4** .

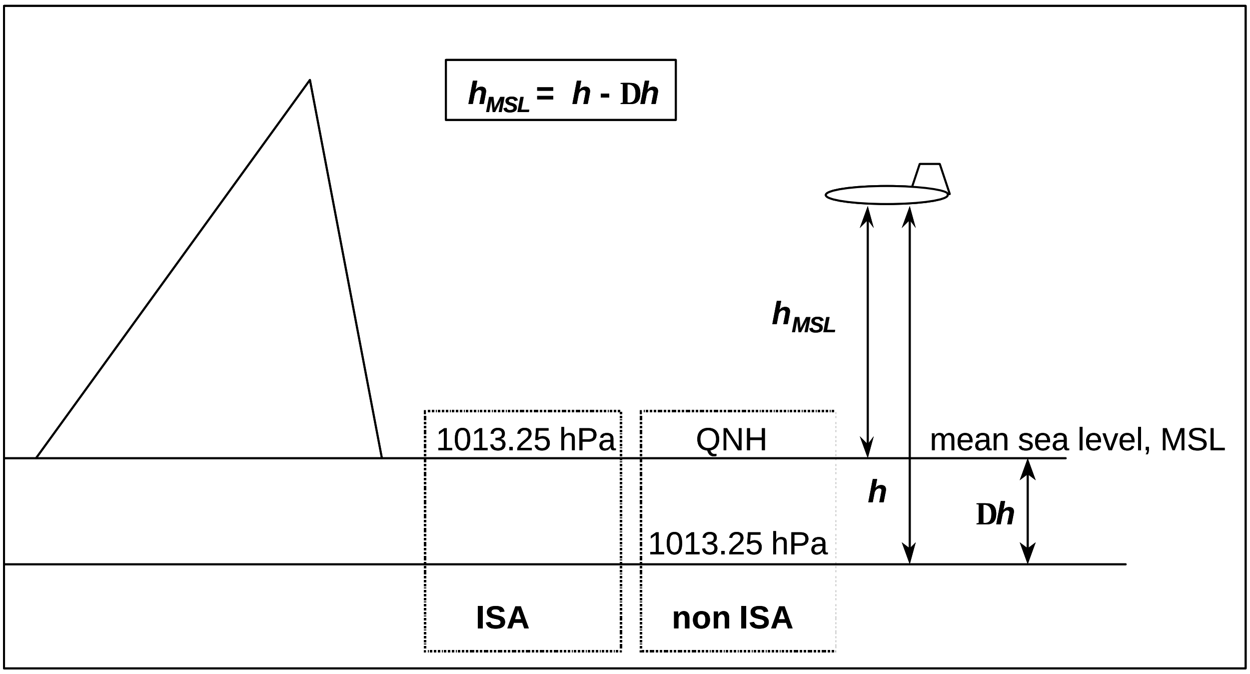


Fig. 1.4: Conversion among different height definitions, pressure, density and temperature in the troposphere

$$\Delta h \approx \Delta H = H = \frac{T_0 + \Delta T}{L}\left(1 - \delta^{\frac{1}{5.25588}}\right)$$

$$\boxed{\Delta h \approx \frac{T_0 + \Delta T}{L}\left(1 - \left(\frac{\mathrm{QNH}}{p_0}\right)^{\frac{1}{5.25588}}\right)}$$

1.3 Aircraft Speed Definitions and Conversions

Indicated Airspeed $\rightarrow$ Calibrated Airspeed $\rightarrow$ Equivalent Airspeed $\rightarrow$ True Airspeed $\rightarrow$ Ground Speed

IAS $\rightarrow$ CAS $\rightarrow$ EAS $\rightarrow$ TAS $\rightarrow$ GS

$V_I \rightarrow V_C \rightarrow V_E \rightarrow V \rightarrow V_G$

Conversion between different aircraft speed definitions

$V_C \rightarrow V_E$:

$$V_E = a_0 \sqrt{5\delta \left\{ \left[\frac{1}{\delta} \left\{ \left[1 + 0.2 \left(\frac{V_C}{a_0} \right)^2 \right]^{3.5} - 1 \right\} + 1 \right]^{\frac{1}{3.5}} - 1 \right\}}$$

or

$$V_E = V_C - \Delta V_C$$

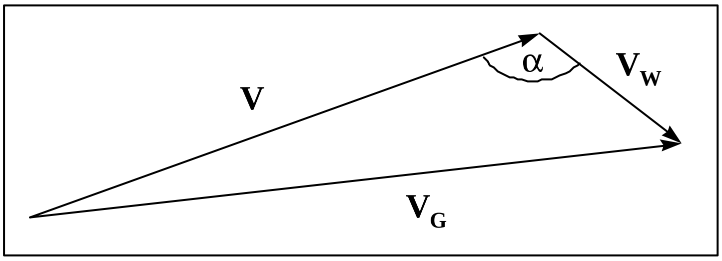
ΔV_C from **Fig. 1.5**

$V_E \rightarrow V$:

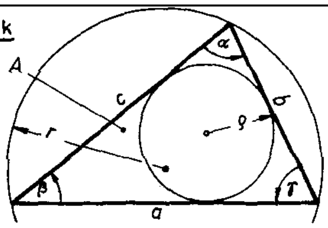
$$V = V_E / \sqrt{\sigma}$$

$V \rightarrow V_G$:

$$V_G = \sqrt{V^2 + V_W{}^2 - 2 \cdot V \cdot V_W \cdot \cos\alpha}$$



Das schiefwinklige Dreieck



Sinus-Satz

$$\sin\alpha : \sin\beta : \sin\gamma = a : b : c$$

$$a = \frac{b}{\sin\beta}\sin\alpha = \frac{c}{\sin\gamma}\sin\alpha$$

$$b = \frac{a}{\sin\alpha}\sin\beta = \frac{c}{\sin\gamma}\sin\beta$$

$$c = \frac{a}{\sin\alpha}\sin\gamma = \frac{b}{\sin\beta}\sin\gamma$$

Cosinus-Satz

$$a^2 = b^2 + c^2 - 2 \cdot bc \cdot \cos\alpha$$

$$b^2 = c^2 + a^2 - 2 \cdot ac \cdot \cos\beta$$

$$c^2 = a^2 + b^2 - 2 \cdot ab \cdot \cos\gamma$$

(GIECK 1981)

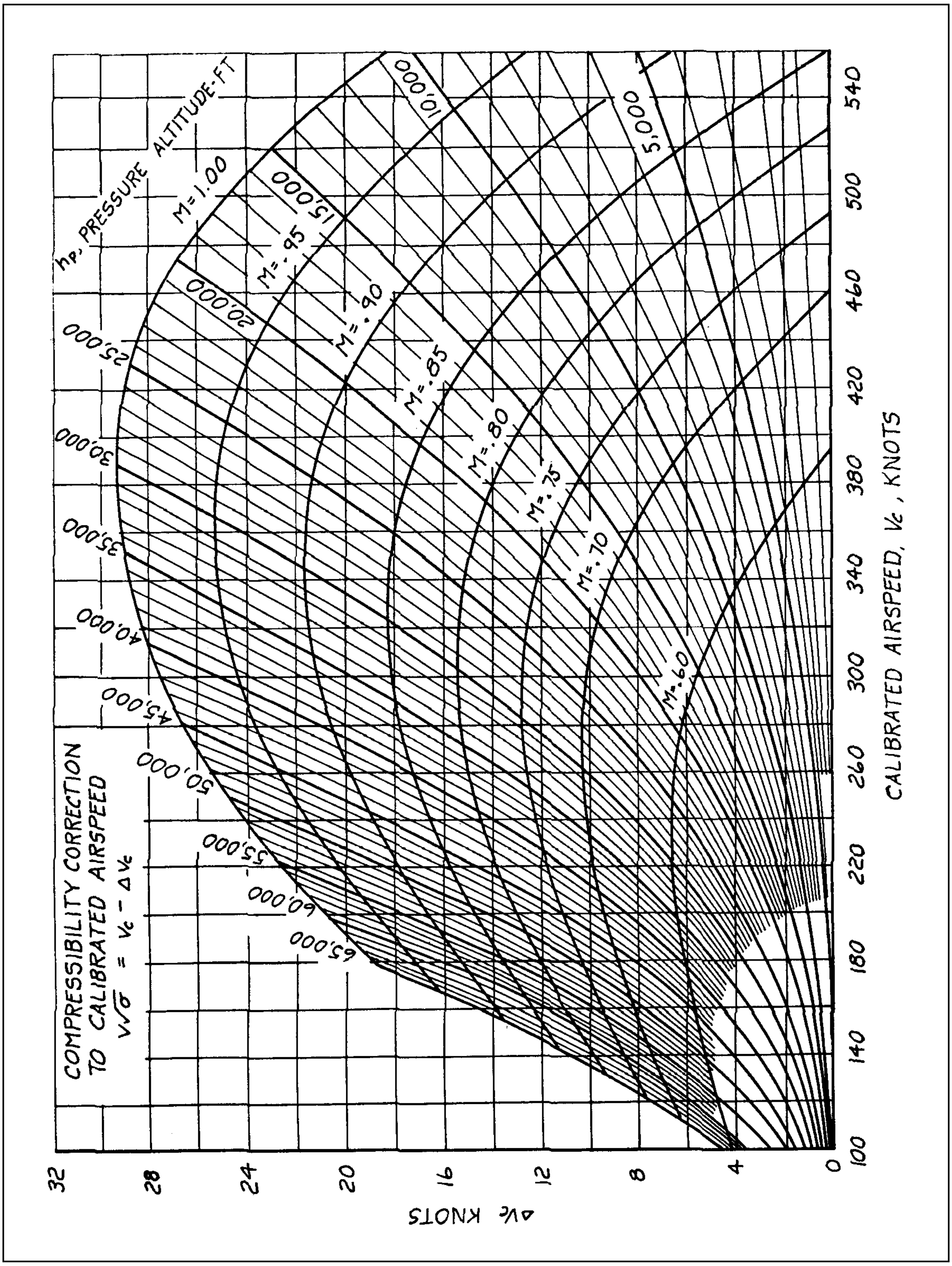


Fig. 1.5: Compressibility correction to calibrated airspeed [17]

$$V_C \rightarrow M: \qquad M=\sqrt{5\left\{\left[\frac{1}{\delta}\left\{\left[1+0.2\left(\frac{V_C}{a_0}\right)^2\right]^{3.5}-1\right\}+1\right]^{\frac{1}{3.5}}-1\right\}}$$

1.4 Air Temperature

Measurements of the outside air temperature are done on board an aircraft with ventilated pitot air thermometers which are shielded against radiation. Since the aircraft is moving relative to the air mass, the temperature sensed by the air thermometer is higher than the ambient air temperature. If the air would be slowed down to stagnation conditions, the measured air temperature would be the stagnation or *total air temperature* T_t (= TAT). The total temperature caused by an adiabatic process is calculated from the static or *outside air temperature* T (= OAT) as a function of Mach number M :

$$\frac{T_T}{T} = 1 + \frac{1}{2}(\gamma - 1) \cdot M^2 = 1 + 0.2 \cdot M^2$$

In the ventilated pitot air thermometer the flow is collected but not totally slowed down to rest. The better the ventilation, the less will the thermometer respond against thermal radiation from the sun or local sources of heat. Ventilation is also necessary to ensure quick response to temperature changes of the air mass. However, ventilation prevents perfect stagnation conditions. The ventilated thermometer can only recover part of the stagnation temperature and shows the *indicated air temperature* T_I (= IAT). A *recovery factor* is defined as

$$k_r = \frac{T_I - T}{T_T - T} \quad .$$

For total recovery k_r = 1. Recovery factors range from about 0.8 to close to 1. A typical value might be k_r = 0.97. The indicated temperature is calculated from

$$\frac{T_I}{T} = 1 + \frac{1}{2} k_r (\gamma - 1) \cdot M^2 = 1 + 0.2 \cdot k_r \cdot M^2 \quad .$$

More often it will be required to calculate the outside air temperature T from the indicated temperature with

$$T = \frac{T_I}{1 + 0.2 \cdot k_r \cdot M^2} \quad .$$

1.5 Rules of Thumb

conversion	method	example
ft → m	ft / 10 * 3	1000 ft ≈ 1000 / 10 * 3 m = 300 m
NM → km	NM * 2 - 10%	1 NM ≈ 1 * 2 - 0.2 km = 1.8 km

1 hPa is equivalent to about 30 ft

3 Aircraft Drag and Drag Power

$$\frac{E}{E_{max}} = \frac{2}{\frac{1}{\overline{C_L}} + \overline{C_L}} \qquad \text{with} \quad \overline{C_L} = \frac{C_L}{C_{L,md}}$$

$C_{L,md}$: lift coefficient in **m**inimum **d**rag flight i.e. flight at E_{max}

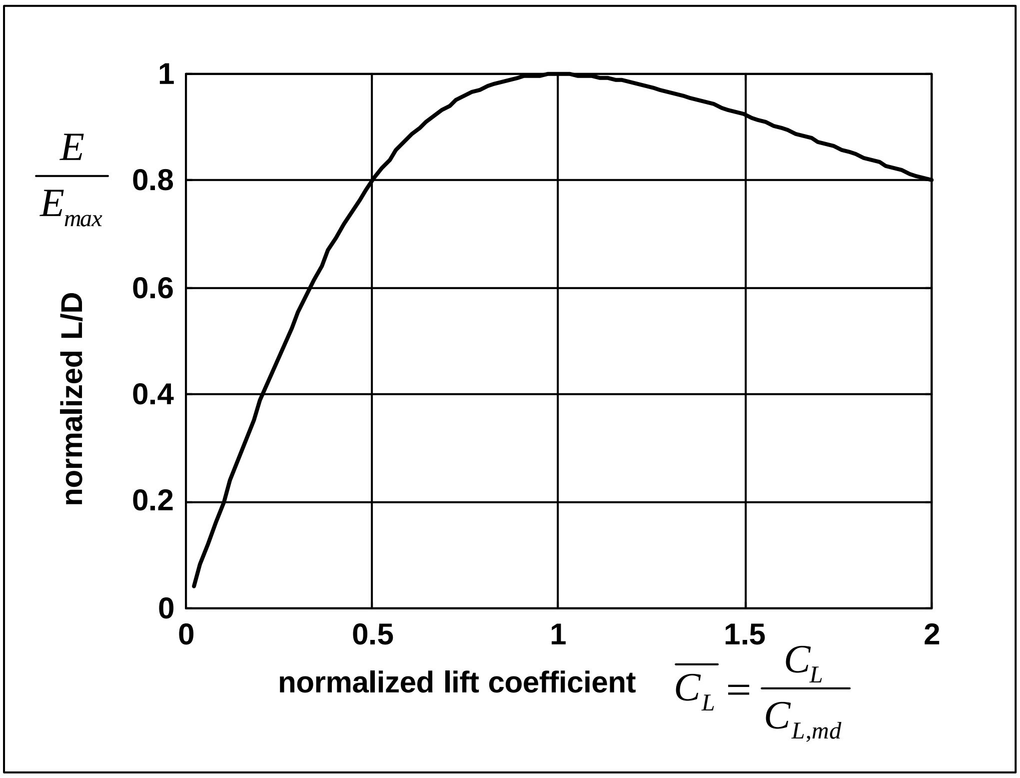


Fig. 3.1: L/D versus lift coefficient in normalized form

4 Powerplant Performance

4.1 Thrust and Efficiency

The product of the thrust of an engine and the forward velocity of the aircraft (relative to the earth, no wind) is equal to the thrust power - also called power available

$$P_T = T\ V \quad .$$

Newton's second law yields the thrust equation (**Fig. 4.1**)

$$T = \dot{m}\left(V_{jet} - V\right) \quad .$$

V_{jet} is the speed of the air leaving the device *relative to the aircraft*. The engine shaft power P_S is related to the thrust power P_T by the efficiency η_P .

$$P_T = \eta_P\ P_S$$

The propulsive efficiency η_P can be defined as

$$\eta_P = \frac{\text{useful power available}}{\text{total power generated}} \quad .$$

The useful power available is power used to push the aircraft forward. The total power generated includes in addition to the power available also the power wasted in the air jet behind the aircraft (**Fig. 4.2**)

$$\eta_P = \frac{T\ V}{T\ V + \frac{1}{2}\dot{m}\left(V_{jet} - V\right)^2} \quad .$$

Substituting the thrust equation yields

$$\eta_P = \frac{\dot{m}\left(V_{jet} - V\right)V}{\dot{m}\left(V_{jet} - V\right)V + \frac{1}{2}\dot{m}\left(V_{jet} - V\right)^2} \quad .$$

Dividing numerator and denominator by $\dot{m}\left(V_{jet} - V\right)V$:

$$h_P = \frac{1}{1 + \frac{1}{2}\frac{\left(V_{jet} - V\right)}{V}} = \frac{1}{1 + \frac{1}{2}\frac{V_{jet}}{V} - \frac{1}{2}} \quad .$$

The *ideal efficiency* of a propulsive device is finally

$$\boxed{\eta_P = \frac{2}{1 + \frac{V_{jet}}{V}}} \quad .$$

Often, the efficiency is expressed by means of the velocity of the jet *relative to the earth* $w = V_{jet} - V$

$$\boxed{\eta_P = \frac{1}{1 + \frac{w/2}{V}}} \quad .$$

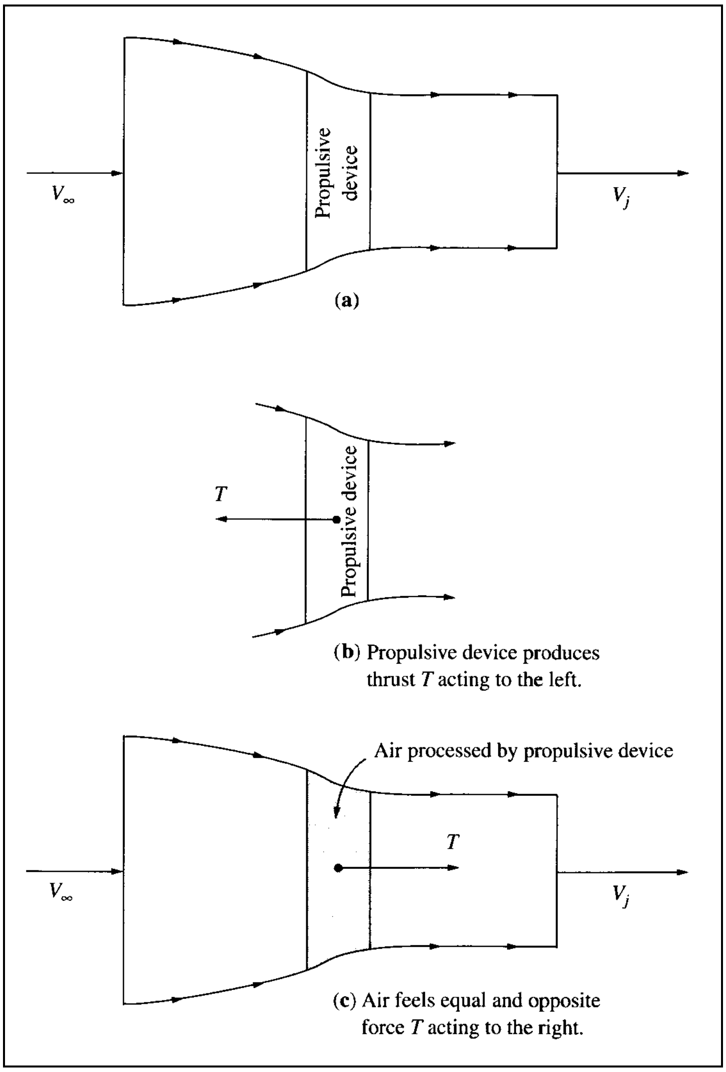


Fig. 4.1:
Reaction principle in propulsion (**ANDERSON 1999**)

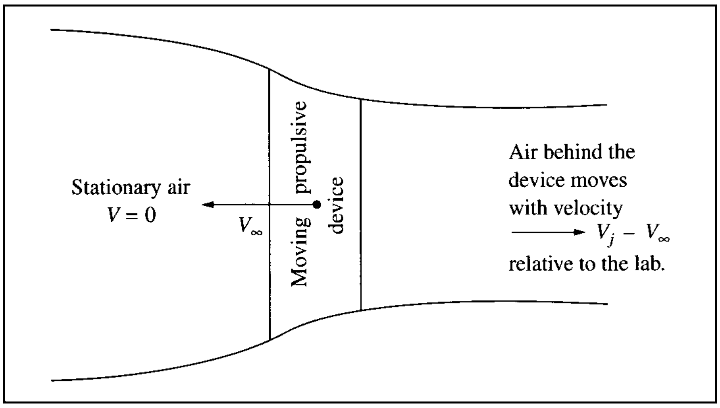


Fig. 4.2:
A propulsive device moving into stationary air with velocity V_∞ (**ANDERSON 1999**)

4.2 Propeller Efficiency

The efficiency calculated above in the form $\eta_P = \dfrac{2}{1+\dfrac{V_{jet}}{V}}$ was called *ideal efficiency* because it does not take into account:

- energy losses due to slipstream rotation,
- blade profile drag,
- non-uniform flow,
- compressibility effects
- propeller blockage due to a fuselage or a nacelle.

For performance calculations *measured propeller efficiencies* have to be used which account for all of the above losses. The efficiency is given as a function of the *advance ratio* (Fortschrittsgrad) of the propeller

$$\boxed{J = \frac{V}{n\,D}} \quad .$$

V is the aircraft speed, n is the propeller speed (number of propeller revolutions per second) and D is the propeller diameter. The advance ratio is proportional to the ratio of aircraft speed and propeller tip speed

$$\left(\frac{V}{r\,\omega}\right)_{tip} = \frac{V}{(D/2)(2\,\pi\,n)} = \frac{V}{D\,\pi\,n} = \frac{J}{\pi} \quad .$$

J is a similarity parameter for propeller performance, in the same category as the Mach number and the Reynolds number. However, propeller performance also depends on the propeller pitch angle β as shown in **Fig. 4.3** and **Fig. 4.4**.

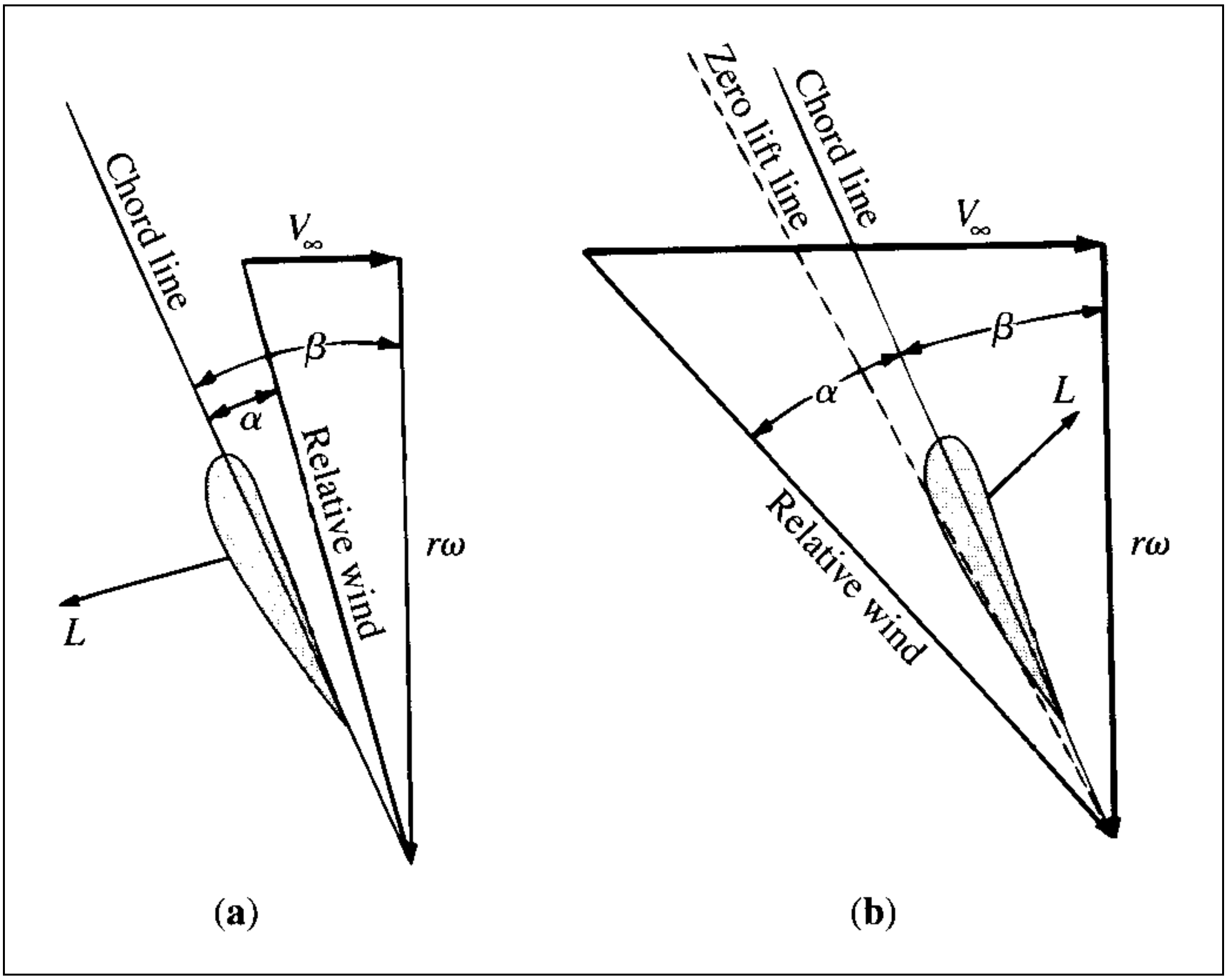


Fig. 4.3: Velocity and relative wind diagrams for a section of a revolving propeller: (a) at low aircraft speeds, (b) at high aircraft speeds (**ANDERSON 1999**)

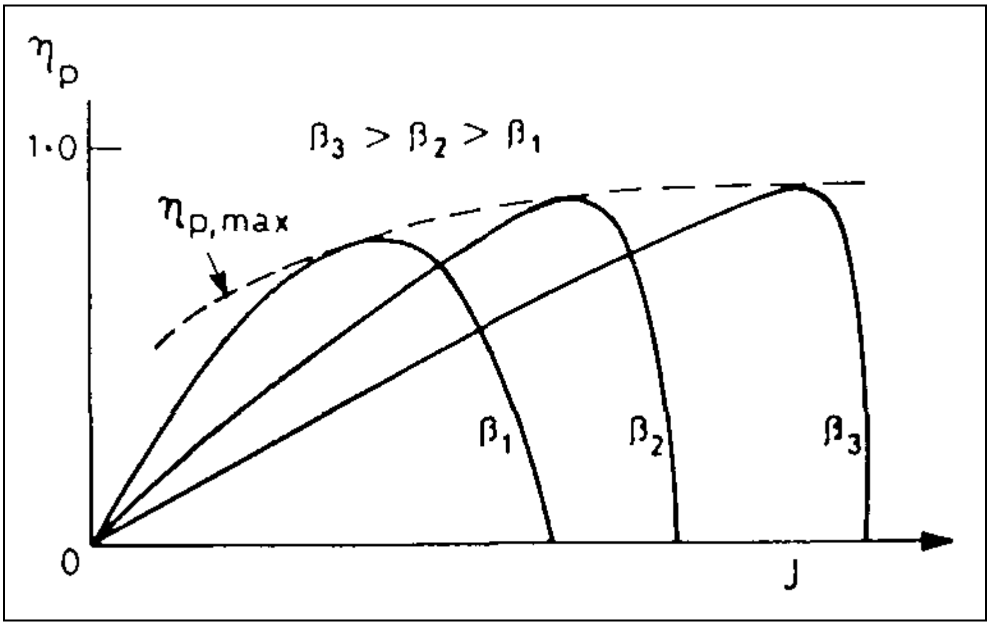


Fig. 4.4: Variation in propeller efficiency η_P with advance ratio J for different pitch angles β [36]

From Fig. 4.3 follows that the propeller has to be twisted. The propeller pitch angle has to be a function of the propeller radius. Propeller charts (like Fig. 4.4) must refer to a pitch angle β at a specific radius as e.g. $0.75 \cdot R$.

For light piston propeller aircraft with a fixed-pitch propeller, the propeller efficiency can be determined from a generic chart presented here as **Fig. 4.5**. The chart shows the efficiency η_P divided by the propeller efficiency at the design point $\left(\eta_P\right)_{design}$ as a function of the advance ratio J divided by the advance ratio at the design point J_{design}

$$\frac{J}{J_{design}} = \frac{V}{V_{design}} \cdot \frac{n_{design}}{n} \quad .$$

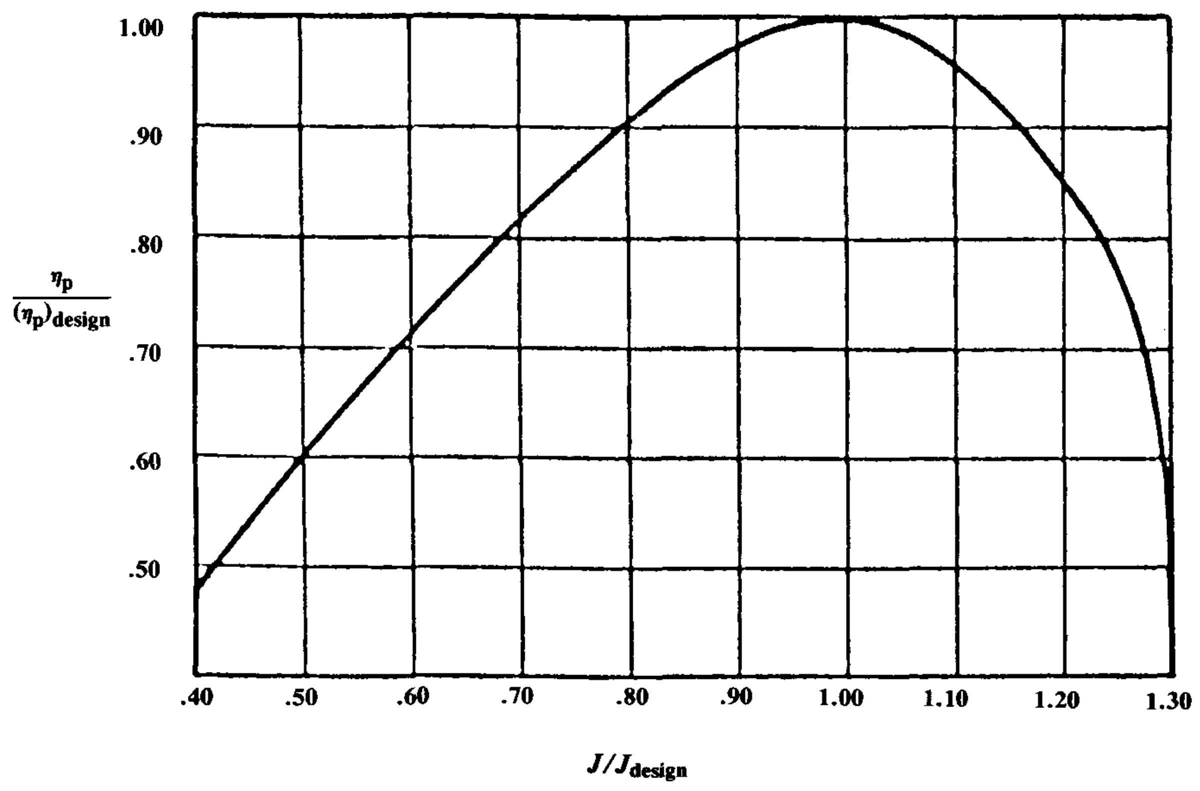


Fig. 4.5: General chart for fixed-pitch propeller efficiency [28]

An example is presented on how the general chart (Fig. 4.5) is used:

A propeller is designed for cruise speeds of 100 kt and 2400 rpm (revolutions per minute) achieving an efficiency $\left(\eta_P\right)_{design}$ = 0.8. What is the efficiency η_P during climb with an aircraft speed of 80 kt and 2500 rpm?

Answer:

$$\frac{J}{J_{design}} = \frac{V}{V_{design}} \cdot \frac{n_{design}}{n} = \frac{80}{100} \cdot \frac{2400}{2500} = 0.768$$

and with Fig. 4.5

$$\eta_P = \left(\eta_P\right)_{design} \cdot \frac{\eta_P}{\left(\eta_P\right)_{design}} = 0.8 \cdot 0.87 = 0.7 \quad .$$

Thrust is calculated from

$$\boxed{T = \frac{\eta_P \cdot P_S}{V}} \quad .$$

The **static thrust** at the beginning of the take-off run can not be calculated from the above equation because at V = 0 also J = 0 and hence η_P = 0. So we have a case 0/0. "As a rough approximation it can be assumed that the static thrust is about equal to the thrust at 50 knots for the fixed-pitch propeller." [28]

5 Level, Climbing and Descending Flight

5.1 Climb and Climb Schedules

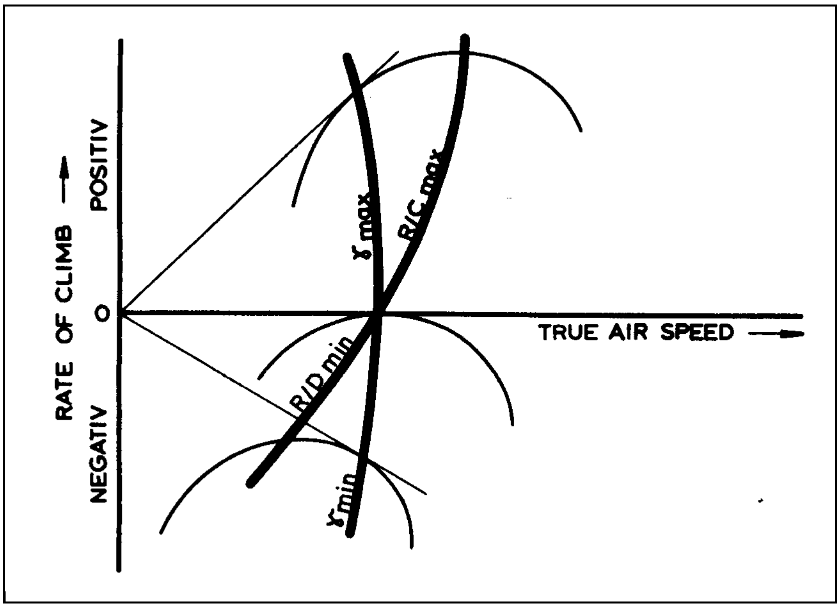


Fig. 5.1:
Maximum rate of climb and maximum climb angle for different climb performance (LUFTHANSA 1988)

For positive climb gradients the speed for best rate of climb is above that speed for best climb angle (**Fig. 5.1**). With decreasing climb performance, the speeds for best angle and best rate of climb come together. Finally, in the range of negative rate of climb, the speed for best angle is above that for best descent rate. Hence, in descending flight, range is maximized with a speed above that for maximum time aloft.

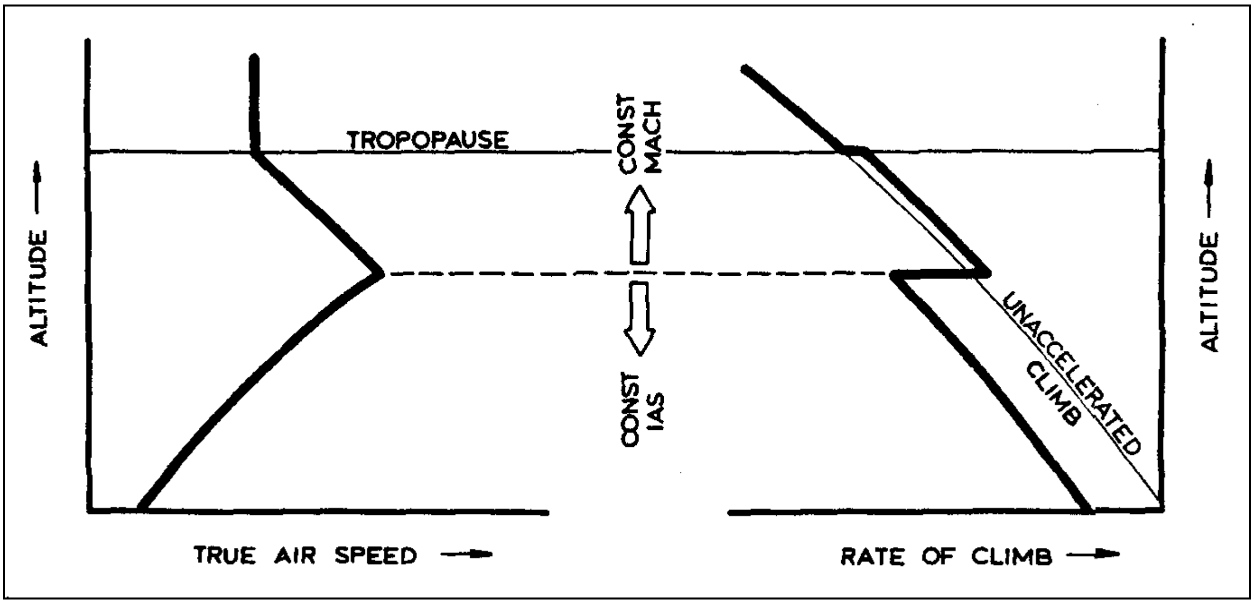


Fig. 5.2: A typical climb schedule and resulting rate of climb (LUFTHANSA 1988)

A typical climb schedule (**Fig. 5.2-left**) consists of three parts:

1. In the lower part of the troposphere, the climb is performed at constant IAS. This is done up to a point where a certain Mach number is reached.
2. It follows a constant Mach number climb in the troposphere results in a decreasing true air speed.
3. In the stratosphere, the continued constant Mach number climb results in a flight at constant true air speed.

In part 1, the climb rate is about 25% below the theoretical values for unaccelerated climb. In part 2, the climb rate is about 9% above the theoretical values for unaccelerated climb. During transition from constant IAS flight to constant Mach number flight, an increase in climb rate of up to 30% can be observed LUFTHANSA 1988. In part 3, climb with constant true air speed results in an unaccelerated climb.

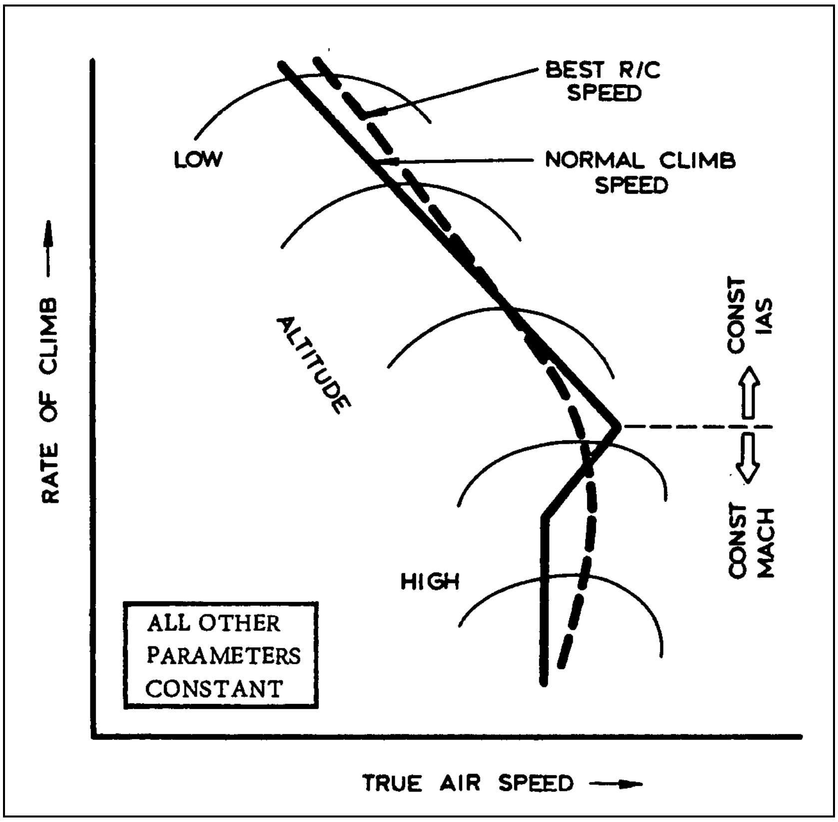


Fig. 5.3:
Influence of altitude on rate of climb (**LUFTHANSA 1988**)

Fig. 5.3 shows the rate of climb as a function of true air speed for different altitudes. With increasing altitude, the thrust decreases at a higher rate than the drag. On account of this, the excess thrust and the rate of climb drop. Fig. 5.3 indicates also the speed for a maximum rate of climb. This optimum climb schedule is approximated with a constant IAS / constant Mach number climb schedule.

Fig. 5.4 shows an increase in the speed for best rate of climb with increasing weight.

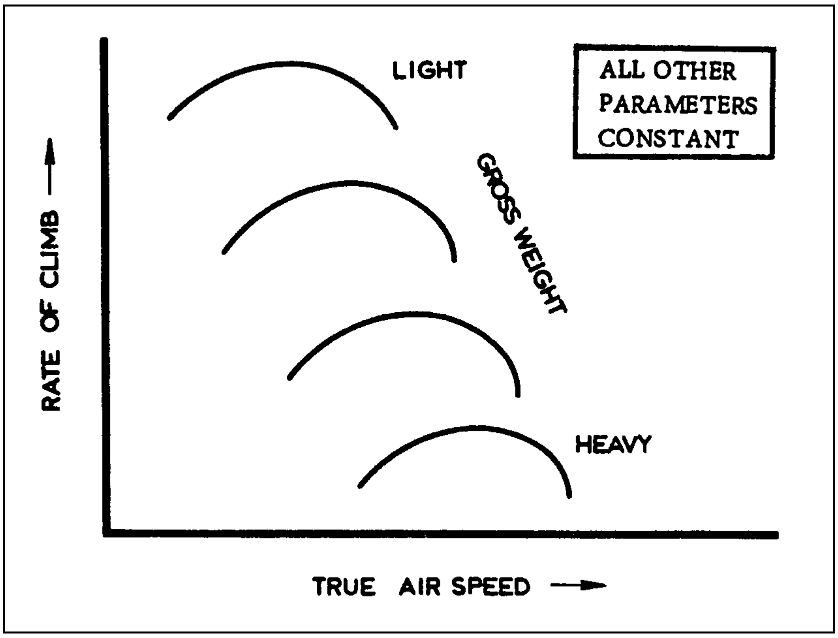


Fig. 5.4:
Influence of aircraft weight on rate of climb (LUFTHANSA 1988)

5.2 Time to Climb

Assume a linear variation of the rate of climb $ROC = V_v$ within small altitude intervals Δh. The time to climb through this altitude interval Δh is Δt. The time to climb to an altitude $h = j \cdot \Delta h$

is $t = \sum_{i=1}^{j} \Delta t_i$ with $\Delta t = \frac{1}{A_i} \cdot \ln\left(\frac{V_{v_{i+1}}}{V_{v_i}}\right)$ and $A_i = \frac{V_{v_{i+1}} - V_{v_i}}{\Delta h} = \frac{V_{v_{i+1}} - V_{v_i}}{h_{i+1} - h_i}$.

A detailed derivation of the Equation for Δt from Chapter 5 Page 23 is given in Appendix A.

5.3 Approximate Climb Calculation

The approximate climb calculation assumes a linear variation of the rate of climb $ROC = V_v$ within the whole climb segment. This assumption is illustrated in **Fig. 5.5**.

$$ROC = ROC_0 - \frac{ROC_0}{h_{abs}} \cdot h = ROC_0 \cdot \left(1 - \frac{h}{h_{abs}}\right) \quad \text{or} \quad \frac{ROC}{ROC_0} = 1 - \frac{h}{h_{abs}}$$

$$t_{CLB} = \frac{1}{A} \cdot \ln\left(\frac{ROC}{ROC_0}\right) \quad \text{with} \quad A = \frac{ROC - ROC_0}{h} = -\frac{ROC_0}{h_{abs}}$$

$$\boxed{t_{CLB} = -\frac{h_{abs}}{ROC_0} \cdot \ln\left(\frac{ROC}{ROC_0}\right)} \quad \text{or} \quad \boxed{t_{CLB} = -\frac{h_{abs}}{ROC_0} \cdot \ln\left(1 - \frac{h}{h_{abs}}\right)} \quad .$$

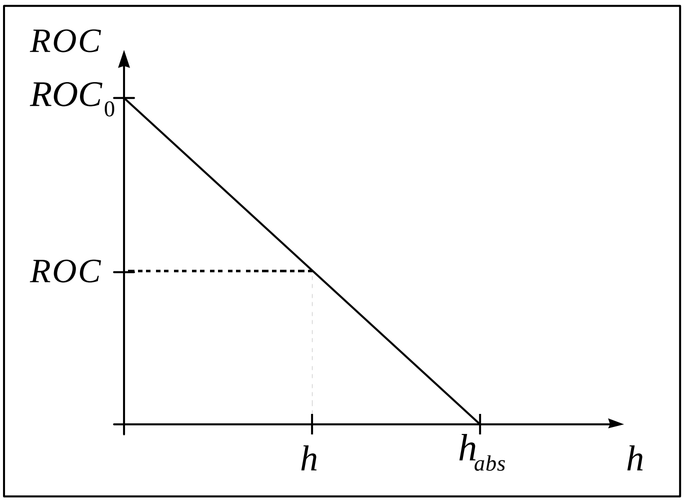


Fig. 5.5:
Assumptions for the approximate climb calculation

6 Stall, Speed Stability, Turning Performance

It might be useful to introduce the standard instruments of light aircraft at this point (**Fig. 6.1**).

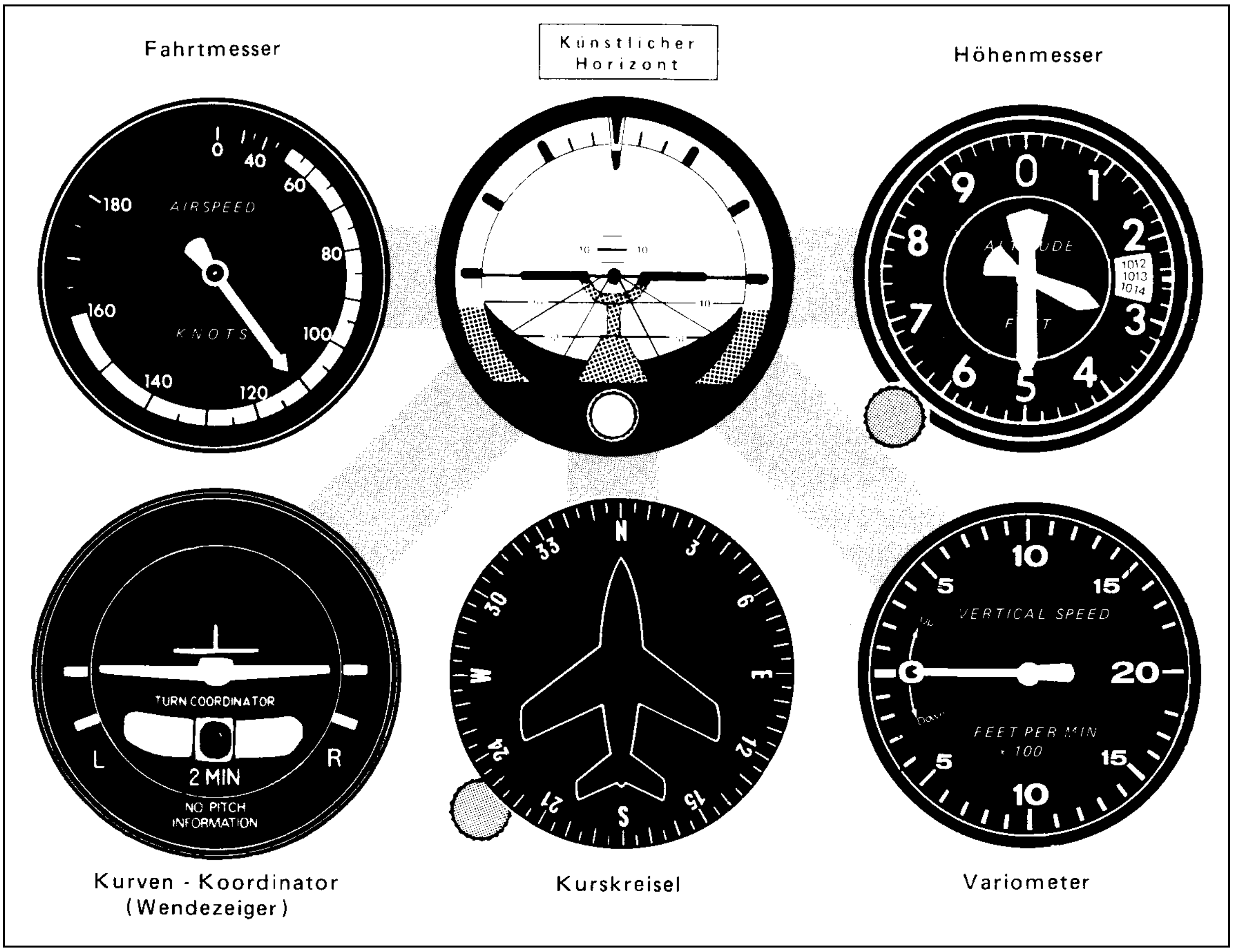


Fig. 6.1: Standard instruments of light aircraft (**KÜHR 1987**)

Table 6.1: Standard instruments and indicated flight parameters

Fahrtmesser	airspeed indicator	V_{IAS}	indicated air speed, IAS
künstlicher Horizont	artificial horizon	Φ, Θ	bank angle, pitch attitude
Höhenmesser	altimeter	h	altitude
Kurven-Koordinator (Wendezeiger)	turn and bank (or slip) indicator	Ω at markings: 3°/s	rate of turn
Kurskreisel	directional giro		heading
Variometer	vertical speed indicator	$dh / dt = V_v = ROC$	vertical speed

References

ANDERSON 1989 ANDERSON, J.D.: *Introduction to Flight*. New York : McGraw-Hill, 1989

ANDERSON 1999 ANDERSON, J.D.: *Aircraft Performance and Design*. Boston : McGraw-Hill, 1989

GIECK 1981 GIECK, K.: *Technische Formelsammlung*. Heilbronn : Gieck, 1981

KÜHR 1987 KÜHR, W.: *Der Privatflugzeugführer - Technik II*. Vol. 3. Bergisch Gladbach : Schiffmann, 1987

RAYMER 1992 RAYMER, D.P.: *Aircraft Design: A Conceptual Approach*. AIAA Education Series, Washington D.C. : AIAA, **1992**

LUFTHANSA 1988 LUFTHANSA CONSULTING (Hrsg.): *Jet Airplane Performance*. 1. Aufl. Köln : Lufthansa Consulting, 1988

Appendix

A Derivations

$$\boxed{H = \frac{r_{earth} \cdot h}{r_{earth} + h}}$$

geopotential height calculated from geometric height

By definition: potential energy = $m \cdot g_0 \cdot H = \int_0^h m \cdot g(y) \cdot dy$ **(1)**

Newton: $F = G \cdot \frac{m \cdot m_{earth}}{r^2} = m \cdot g(h)$ $r = r_{earth} + h$

$$g(h) = g_0 \left(\frac{r_{earth}}{r_{earth} + h} \right)^2 \qquad \textbf{(2)}$$

from (1) and (2): $H = \frac{1}{g_0} \int_0^h g_0 \left(\frac{r_{earth}}{r_{earth} + y} \right)^2 dy$

$$H = r_{earth}{}^2 \int_0^h \frac{1}{\left(r_{earth} + y\right)^2} dy$$

$$H = -r_{earth}{}^2 \left[\frac{1}{r_{earth} + y} \right]_0^h = -r_{earth}{}^2 \left[\frac{1}{r_{earth} + h} - \frac{1}{r_{earth}} \right] = -r_{earth}{}^2 \left[\frac{r_{earth} - \left(r_{earth} + h\right)}{\left(r_{earth} + h\right) \cdot r_{earth}} \right]$$

$$H = \frac{r_{earth} \cdot h}{r_{earth} + h}$$

$$\boxed{H = h_p \cdot \frac{T_0 + \Delta T}{L}}$$

geopotential height calculated from pressure height in the troposhere

$$p = p_0 \cdot \left(1 - \frac{L}{T_0 + \Delta T} \cdot H \right)^{5.25588}$$

for an atmosphere with L, R, g_0 from ISA but with **(1)**

$T(h = 0) = T_0 + \Delta T$

$$p = p_0 \cdot \left(1 - \frac{L}{T_0} \cdot h_p\right)^{5.25588}$$ by definition **(2)**

$$p = p_0 \cdot \left(1 - \frac{L}{T_0 + \Delta T} \cdot H\right)^{5.25588} = p_0 \cdot \left(1 - \frac{L}{T_0} \cdot h_p\right)^{5.25588}$$ from (1) and (2)

$$\left(1 - \frac{L}{T_0 + \Delta T} \cdot H\right)^{5.25588} = \left(1 - \frac{L}{T_0} \cdot h_p\right)^{5.25588}$$

$$\frac{L}{T_0 + \Delta T} \cdot H = \frac{L}{T_0} \cdot h_p$$

$$H = h_p \cdot \frac{T_0 + \Delta T}{T_0}$$

$$\boxed{\frac{E}{E_{max}} = \frac{2}{\frac{1}{\overline{C_L}} + \overline{C_L}}}$$ with $\overline{C_L} = \frac{C_L}{C_{L,md}}$

$$\frac{C_D}{C_{D,md}} = \frac{C_{D0} + \frac{C_L^{\ 2}}{\pi \cdot A \cdot e}}{C_{D0} + \frac{C_{L,md}^{\ 2}}{\pi \cdot A \cdot e}}$$

$C_{D,md}$: minimum drag flight i.e. flight at E_{max}

$$\frac{C_D}{C_{D,md}} = \frac{C_{D0} + \frac{C_L^{\ 2}}{\pi \cdot A \cdot e}}{2 \cdot C_{D0}} = \frac{1}{2} + \frac{C_L^{\ 2}}{2 \cdot C_{D0} \cdot \pi \cdot A \cdot e} = \frac{1}{2}\left(1 + \frac{C_L^{\ 2}}{C_{D0} \cdot \pi \cdot A \cdot e}\right) = \frac{1}{2}\left(1 + \frac{C_L^{\ 2}}{\frac{C_{L,md}^{\ 2}}{\pi \cdot A \cdot e} \cdot \pi \cdot A \cdot e}\right)$$

$$= \frac{1}{2}\left(1 + \frac{C_L^{\ 2}}{C_{L,md}^{\ 2}}\right) = \frac{1}{2}\left(1 + \left(\frac{C_L}{C_{L,md}}\right)^2\right) = \frac{1}{2}\left(1 + \overline{C_L}^{\,2}\right)$$

$$\frac{E}{E_{max}} = \frac{\frac{C_L}{C_D}}{\frac{C_{L,md}}{C_{D,md}}} = \frac{C_{D,md}}{C_D} \cdot \frac{C_L}{C_{L,md}} = \frac{2}{1 + \overline{C_L}^{\,2}} \cdot \overline{C_L} = \frac{2}{\frac{1}{\overline{C_L}} + \overline{C_L}}$$

$$\boxed{\Delta t = \frac{1}{A_i} \cdot \ln\left(\frac{V_{v_{i+1}}}{V_{v_i}}\right)}$$

A more detailed derivation of Equation [5.3-25] from chapter 5 page 23.

Integrating time to climb from ROC:

$$V_v = \frac{dh}{dt} \qquad dt = \frac{1}{V_v} dh \qquad t = \int_{h_i}^{h_{i+1}} \frac{1}{V_v(h)} dh$$

Assuming a linear variation of ROC within interval *i*:

$$V_v = \frac{V_{v_{i+1}} - V_{v_i}}{\Delta h} \cdot (h - h_i) + V_{v_i} \qquad A_i = \frac{V_{v_{i+1}} - V_{v_i}}{\Delta h} = \frac{V_{v_{i+1}} - V_{v_i}}{h_{i+1} - h_i}$$

Introducing a new variable:

$$h' = h - h_i$$

$$t = \int_{h_i}^{h_{i+1}} \frac{1}{A_i \cdot (h - h_i) + V_{v_i}} dh$$

Substitution:

$$t = \int_{h'_i}^{h'_{i+1}} \frac{1}{A_i \cdot h' + V_{v_i}} \cdot \frac{dh}{dh'} dh' \qquad t = \int_{h'_i}^{h'_{i+1}} \frac{1}{A_i \cdot h' + V_{v_i}} dh'$$

Solving the integral:

$$t = \frac{1}{A_i} \cdot \left[\ln\left(A_i \cdot h' + V_{v_i}\right)\right]_{h'_i}^{h'_{i+1}} = \frac{1}{A_i} \cdot \left[\ln\left(A_i \cdot h' + V_{v_i}\right)\right]_{h_i - h_i}^{h_{i+1} - h_i}$$

$$t = \frac{1}{A_i} \cdot \left[\ln\left(V_{v_{i+1}}\right) - \ln\left(V_{v_i}\right)\right]$$

$$t = \frac{1}{A_i} \cdot \ln\left(\frac{V_{v_{i+1}}}{V_{v_i}}\right)$$

This time t - also called Δt - is the time required to climb from an altitude h_i to an altitude h_{i+1}.

$$\Delta t = \frac{1}{A_i} \cdot \ln\left(\frac{V_{v_{i+1}}}{V_{v_i}}\right)$$