Dieter Scholz

Corrections

of Selected Chapters of the Book

McLean, Donald: "Automatic Flight Control Systems". New York : Prentice Hall, 1990.

2022

Preface

The book

McLean, Donald: "Automatic Flight Control Systems". New York : Prentice Hall, 1990.

has more errors than acceptable. I offered the publisher my corrections in 1992, but the publisher had no money for a second edition. This is apparently the same situation 30 years later.

Long ago, I also met McLean at a conference and discussed the matter with him. Subsequently, I also sent him my corrections, knowing that this would not change the situation either.

Books are published without peer review process. This can be problematic as we see in this case. In order to set the scientific record straight (as much as I can contribute to it), I offered my stundents the corretions already in 2003. For all other readers with an interest the corrections are now offered in one archived PDF file.

Except form the many errors, the book by Professor Donald McLean is very valuable and it would be a loss not to have it.

Dieter Scholz, 2022

PRENTICE HALL INTERNATIONAL
SERIES IN SYSTEMS AND
CONTROL ENGINEERING
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Automatic Flight Control Systems

DONALD McLEAN

MCLEAN, Donald: *Automatic Flight Control Systems*. New York : Prentice Hall, 1990.

ISBN: 0-13-054008-0
First Edition
The book is presently (2003) out of print.

From the text book only selected Chapters are used in the lecture FM2 at
Hamburg University of Applied Science:

Note: Some Chapters are only partially copied.

1

Aircraft Flight Control

1.1 INTRODUCTION

Whatever form a vehicle may take, its value to its user depends on how effectively it can be made to proceed in the time allowed on a precisely controllable path between its point of departure and its intended destination. That is why, for instance, kites and balloons find only limited application in modern warfare. When the motion of any type of vehicle is being studied it is possible to generalize so that the vehicle can be regarded as being fully characterized by its velocity vector. The time integral of that vector is the path of the vehicle through space (McRuer *et al.*, 1973). The velocity vector, which may be denoted as $\dot{\mathbf{x}}$, is affected by the position, $\mathbf{x}$, of the vehicle in space by whatever kind of control, $\mathbf{u}$, can be used, by any disturbance, $\boldsymbol{\xi}$, and by time, t. Thus, the motion of the vehicle can be represented in the most general way by the vector differential equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\xi}, t) \tag{1.1}$$

where $\mathbf{f}$ is some vector function. The means by which the path of any vehicle can be controlled vary widely, depending chiefly on the physical constraints which obtain. For example, everyone knows that a locomotive moves along the rails of the permanent way. It can be controlled only in its velocity; it cannot be steered, because its lateral direction is constrained by the contact of its wheel rims on the rails. Automobiles move over the surface of the earth, but with both speed and direction being controlled. Aircraft differ from locomotives and automobiles because they have six degrees of freedom: three associated with angular motion about the aircraft's centre of gravity and three associated with the translation of the centre of gravity.[1] Because of this greater freedom of motion, aircraft control problems are usually more complicated than those of other vehicles.

Those qualities of an aircraft which tend to make it resist any change of its velocity vector, either in its direction or its magnitude, or in both, are what constitutes its *stability*. The ease with which the velocity vector may be changed is related to the aircraft's quality of *control*. It is *stability* which makes possible the maintenance of a steady, unaccelerated flight path; aircraft manoeuvres are effected by *control*.

Of itself, the path of any aircraft is never stable; aircraft have only neutral stability in heading. Without control, aircraft tend to fly in a constant turn. In order to fly a straight and level course continuously-controlling corrections must be made, either through the agency of a human pilot, or by means of an *automatic* *flight control system* (AFCS). In aircraft, such AFCSs employ feedback control to achieve the following benefits:

1. The speed of response is better than from the aircraft without closed loop control.
2. The accuracy in following commands is better.
3. The system is capable of suppressing, to some degree, unwanted effects which have arisen as a result of disturbances affecting the aircraft's flight.

However, under certain conditions such feedback control systems have a tendency to oscillate; the AFCS then has poor stability. Although the use of high values of gain in the feedback loops can assist in the achievement of fast and accurate dynamic response, their use is invariably inimical to good stability. Hence, designers of AFCSs are obliged to strike an acceptable, but delicate, balance between the requirements for stability and for control.

The early aeronautical experimenters hoped to make flying easier by providing 'inherent' stability in their flying machines. What they tried to provide was a basic, self-restoring property of the airframe without the active use of any feedback. A number of them, such as Cayley, Langley and Lilienthal, discovered how to achieve longitudinal static stability with respect to the relative wind, e.g. by setting the incidence of the tailplane at some appropriate value. Those experimenters also discovered how to use wing dihedral to achieve lateral static stability. However, as aviation has developed, it has become increasingly evident that the motion of an aircraft designed to be inherently very stable, is particularly susceptible to being affected by atmospheric turbulence. This characteristic is less acceptable to pilots than poor static stability.

It was the great achievement of the Wright brothers that they ignored the attainment of inherent stability in their aircraft, but concentrated instead on making it controllable in moderate weather conditions with average flying skill. So far in this introduction, the terms dynamic and static stability have been used without definition, their imprecise sense being left to the reader to determine from the text. There is, however, only one dynamic property – stability – which can be established by any of the theories of stability appropriate to the differential equations being considered. However, in aeronautical engineering, the two terms are still commonly used; they are given separate specifications for the flying qualities to be attained by any particular aircraft. When the term static stability is used, what is meant is that if a disturbance to an aircraft causes the resulting forces and moments acting on the aircraft to tend initially to return the aircraft to the kind of flight path for which its controls are set, the aircraft can be said to be statically stable. Some modern aircraft are not capable of stable equilibrium – they are statically unstable. Essentially, the function of static stability is to recover the original speed of equilibrium flight. This does not mean that the initial flight path is resumed, nor is the new direction of motion necessarily the same as the old. If, as a result of a disturbance, the resulting forces and moments do not tend initially to restore the aircraft to its former equilibrium flight path, but leave it in its disturbed state, the aircraft is neutrally stable. If it tends initially to deviate further from its equilibrium flight path, it is statically unstable. When an aircraft is put in a state of equilibrium by the action of the pilot adjusting the controls, it is said to be trimmed. If, as a result of a disturbance, the aircraft tends to return eventually to its equilibrium flight path, and remains at that position, for some time, the aircraft is said to be dynamically stable. Thus, dynamic stability governs how an aircraft recovers its equilibrium after a disturbance. It will be seen later how some aircraft may be statically stable, but are dynamically unstable, although aircraft which are statically unstable will be dynamically unstable.

1.2 CONTROL SURFACES

Every aeronautical student knows that if a body is to be changed from its present state of motion then external forces, or moments, or both, must be applied to the body, and the resulting acceleration vector can be determined by applying Newton's Second Law of Motion. Every aircraft has control surfaces or other means which are used to generate the forces and moments required to produce the accelerations which cause the aircraft to be steered along its three-dimensional flight path to its specified destination.

A conventional aircraft is represented in Figure 1.1. It is shown with the usual control surfaces, namely elevator, ailerons, and rudder. Such conventional aircraft have a fourth control, the change in thrust, which can be obtained from the engines. Many modern aircraft, particularly combat aircraft, have considerably more control surfaces, which produce additional control forces or moments. Some of these additional surfaces and motivators include horizontal and vertical canards, spoilers, variable cambered wings, reaction jets, differentially operating horizontal tails and movable fins. One characteristic of flight control is that the required motion often needs a number of control surfaces to be used simultaneously. It is shown later in this book that the use of a single control surface always produces other motion as well as the intended motion. When more than one control surface is deployed simultaneously, there often results

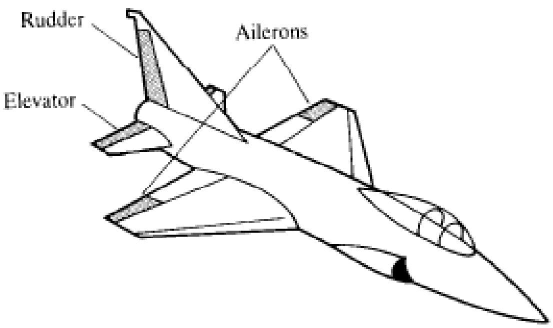


Figure 1.1 Conventional aircraft.

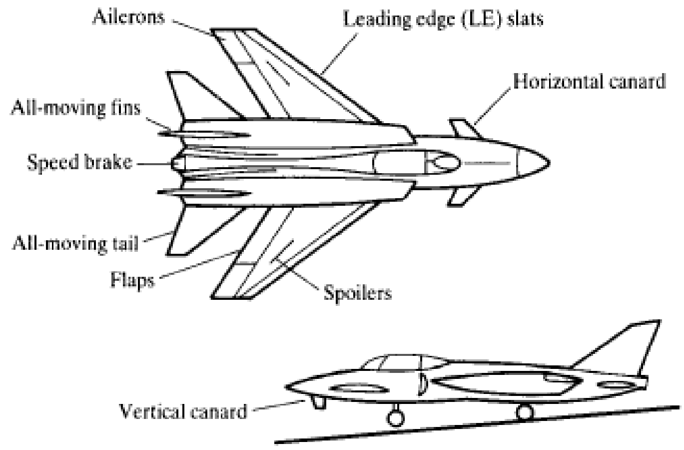


Figure 1.2 A proposed control configured vehicle.

considerable coupling and interaction between motion variables. It is this physical situation which makes AFCS design both fascinating and difficult. When these extra surfaces are added to the aircraft configuration to achieve particular flight control functions, the aircraft is described as a 'control configured vehicle' (CCV). A sketch of a proposed CCV is illustrated in Figure 1.2 in which there are shown a number of extra and unconventional control surfaces. When such extra controls are provided it is not to be supposed that the pilot in the cockpit will have an equal number of extra levers, wheels, pedals, or whatever, to provide the appropriate commands. In a CCV such commands are obtained directly from an AFCS and the pilot has no direct control over the deployment of each individual surface. The AFCS involved in this activity are said to be *active control technology* systems. The surfaces are moved by actuators which are signalled electrically (fly-by-wire) or by means of fibre optic paths (fly-by-light). But, in a conventional aircraft, the pilot has direct mechanical links to the surfaces, and how he commands the deflections, or changes, he requires from the controls is by means of what are called the *primary flying controls*.

1.3 PRIMARY FLYING CONTROLS

In the UK, it is considered that what constitutes a flight control system is an arrangment of all those control elements which enable controlling forces and moments to be applied to the aircraft. These elements are considered to belong to three groups: pilot input elements, system output elements and intervening linkages and elements.

The primary flying controls are part of the flight control system and are defined as the input elements moved directly by a human pilot to cause an operation of the control surfaces. The main primary flying controls are pitch control, roll control and yaw control. The use of these flight controls affects motion principally about the transverse, the longitudinal, and the normal axes respectively, although each may affect motion about the other axes. The use of thrust control via the throttle levers is also effective, but its use is primarily governed by considerations of engine management. Figure 1.3 represents the cockpit layout of a typical, twin engined, general aviation aircraft. The yoke is the primary flying control used for pitch and roll control. When the yoke is pulled towards, or pushed away from, the pilot the elevator is moved correspondingly. When the yoke is rotated to the left or the right, the ailerons of the aircraft are moved. Yaw control is effected by means of the pedals, which a pilot pushes left or right with his feet to move the rudder. In the kind of aircraft with the kind of cockpit illustrated here, the link between these primary flying controls and the control surfaces is by means of cables and pulleys. This means that the aerodynamic forces acting on the control surfaces have to be countered directly by the pilot. To maintain a control surface at a fixed position for any period of time means that the pilot must maintain the required counterforce, which can be very difficult and fatiguing to sustain. Consequently, all aircraft have trim wheels (see Figure 1.3) which the pilot adjusts until the command, which he has set initially on his primary flying control, is set on the control surface and the pilot is then relieved of the need to sustain the force. There are trim wheels for pitch, roll and yaw (which is sometimes referred to as 'nose trim').

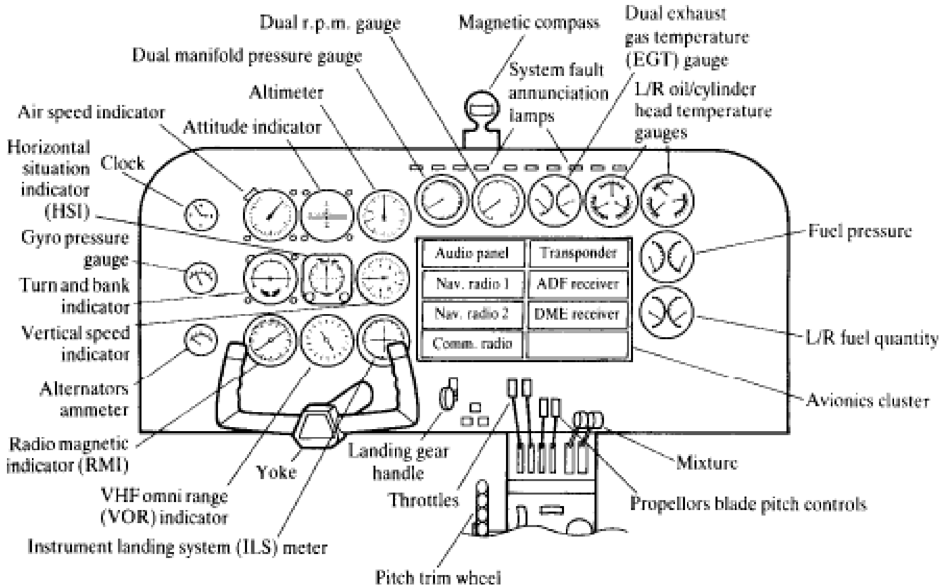


Figure 1.3 Cockpit layout.

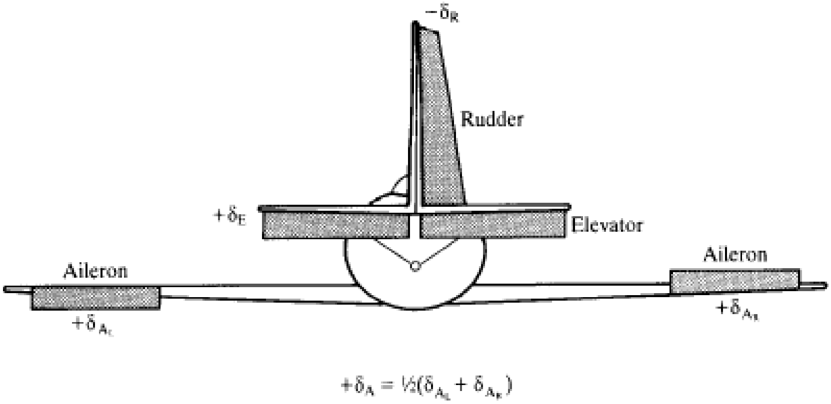


Figure 1.4 Control surface deflection conventions.

In large transport aircraft, or fast military aircraft, the aerodynamic forces acting on the control surfaces are so large that it is impossible for any human pilot to supply or sustain the force required. Powered flying controls are then used. Usually the control surfaces are moved by means of mechanical linkages driven by electrohydraulic actuators. A number of aircraft use electrical actuators, but there are not many such types. The command signals to these electrohydraulic actuators are electrical voltages supplied from the controller of an AFCS, or directly from a suitable transducer on the primary flying control itself. By providing the pilot with power assistance, so that the only force he needs to produce is a tiny force, sufficient to move the transducer, it has been found necessary to provide artificial feel so that some force, representing what the aircraft is doing, is produced on the primary flying control. Such forces are cues to a pilot and are essential to his flying the aircraft successfully. The conventions adopted for the control surface deflections are shown in Figure 1.4.

In the event of an electrical or hydraulic failure such a powered flying control system ceases to function, which would mean that the control surface could not be moved: the aircraft would therefore be out of control. To prevent this occurring, most civilian and military aircraft retain a direct, but parallel, mechanical connection from the primary flying control to the control surface which can be used in an emergency. When this is done the control system is said to have 'manual reversion'. Fly-by-wire (and fly-by-light) aircraft have essentially the same kind of flight control system, but are distinguished from conventional aircraft by having no manual reversion. To meet the emergency situation, when failures occur in the system, fly-by-wire (FBW) aircraft have flight control systems which are triplicated, sometimes quadruplicated, to meet this stringent reliability requirement.

With FBW aircraft and CCVs it has been realized that there is no longer a direct relationship between the pilot's command and the deflection, or even the use, of a particular control surface. What the pilot of such aircraft is commanding from the AFCS is a particular manoeuvre. When this was understood, and when

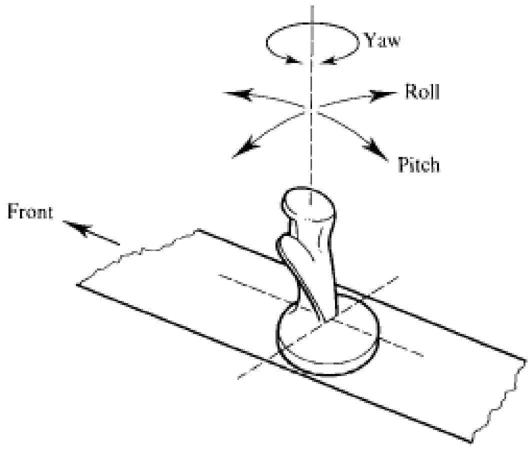


Figure 1.5 Side arm controller.

the increased complexity of flying was taken into account, it was found that the provision of a yoke or a stick to introduce commands was unnecessary and inconvenient. Modern aircraft are being provided with side arm controllers (see Figure 1.5) which provide signals corresponding to the forces applied by the pilot. Generally, these controllers do not move a great deal, but respond to applied force. By using such controllers a great deal of cockpit area is made available for the growing number of avionics displays which modern aircraft require.

1.4 FLIGHT CONTROL SYSTEMS

In addition to the control surfaces which are used for steering, every aircraft contains motion sensors which provide measures of changes in motion variables which occur as the aircraft responds to the pilot's commands or as it encounters some disturbance. The signals from these sensors can be used to provide the pilot with a visual display, or they can be used as feedback signals for the AFCS. Thus, the general structure of an AFCS can be represented as the block schematic of Figure 1.6. The purpose of the controller is to compare the commanded motion with the measured motion and, if any discrepancy exists, to generate, in accordance with the required control law, the command signals to the actuator to produce the control surface deflections which will result in the correct control force or moment being applied. This, in turn, causes the aircraft to respond appropriately so that the measured motion and commanded motion are finally in correspondence. How the required control law can be determined is one of the principal topics of this book.

Whenever either the physical or abstract attributes of an aircraft, and its motion sensing and controlling elements, are considered in detail, their effects are so interrelated as almost to preclude discussion of any single aspect of the system,

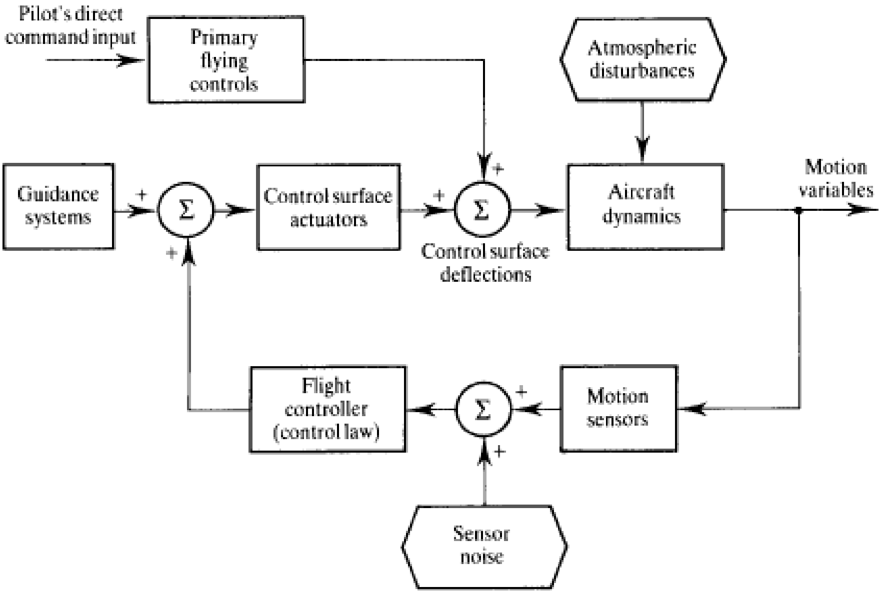


Figure 1.6 General structure of an AFCS.

without having to treat most of the other aspects at the same time. It is helpful, therefore, to define here, albeit somewhat broadly, the area of study upon which this book will concentrate.

1. The development of forces and moments for the purpose of establishing an equilibrium state of motion (operating point) for an aircraft, and for the purpose of restoring a disturbed aircraft to its equilibrium state, and regulating within specific limits the departure of the aircraft's response from the operating point, are regarded here as constituting *flight control*.
2. Regulating the aircraft's response is frequently referred to as *stabilization*.
3. Guidance is taken to mean the action of determining the course and speed to be followed by the aircraft, relative to some reference system. Flight control systems act as interfaces between the guidance systems and the aircraft being guided in that the flight control system receives, as inputs from the guidance systems, correction commands, and provides, as outputs, appropriate deflections of the necessary control surfaces to cause the required change in the motion of the aircraft (Draper, 1981). For this control action to be effective, the flight control system must ensure that the whole system has adequate stability.

   If an aircraft is to execute commands properly, in relation to earth coordinates, it must be provided with information about the aircraft's orientation so that right turn, left turn, up, down, roll left, roll right, for example, are related to the airborne geometrical reference. For about sixty years, it has been common practice to provide aircraft with reference coordinates for control and stabilization by means of gyroscopic instruments. The bank and climb indicator, for example, effectively provides a horizontal reference plane, with an accuracy of a few degrees, and is as satisfactory today for the purposes of control as when it was first introduced. Similarly, the turn indicator, which shows the aircraft's turning left or right, to about the same accuracy, is also a gyroscopic instrument and the use of signals from both these devices, as feedback signals for an AFCS, is still effective and valid. However, the use of conventional gyroscopic instruments in aircraft has fundamental limitations which lie in the inherent accuracy of indication, which is to within a few degrees only, and also in the inherent drift rates, of about ten degrees per hour. Such instruments are unsuitable for present-day navigation, which requires that the accumulated error in distance for each hour of operation, after an inertial fix, be not greater than 1.5 km. An angle of one degree between local gravitational directions corresponds to a distance on the earth's surface of approximately 95 km. Consequently, special motion sensors, such as ring laser gyros, NMR gyros, strap-down, force-balance accelerometers, must be used in modern flight control systems.

Because this book is concerned with control, rather than guidance, it is more convenient to represent the motion of aircraft in a system of coordinates which is fixed in the aircraft and moves with it. By doing this, the coordinate transformations generally required to obtain the aircraft's motion in some other coordinate system, such as a system fixed in the earth, can be avoided. When the origin of such a body-fixed system of coordinates is fixed at the centre of gravity of the aircraft, which is in an equilibrium (or trimmed) state of motion along a nominal flight path, then, when only small perturbations of the aircraft's motion about this equilibrium state are considered, the corresponding equations of motion can be linearized. Since many flight control problems are of very short duration (5–20 seconds), the coefficients of these equations of motion can be regarded as constant, so that transfer functions can sometimes be conveniently used to describe the dynamics of the aircraft. However, it must be remembered that a notable feature of an aircraft's dynamic response is how it changes markedly with forward speed, height, and the aircraft's mass. Some of the most difficult problems of flight control occurred with the introduction of jet propulsion, the consequent expansion of the flight envelope of such aircraft, and the resulting changes in configuration, most notable of which were the use of swept wings, of very short span and greatly increased wing loading, and the concentrated mass of the aircraft being distributed in a long and slender fuselage. In aircraft of about 1956 these changes led to marked deficiences in the damping of the classical modes of aircraft motion, namely the short period mode of the aircraft's longitudinal motion, and the Dutch roll mode of its lateral motion. Other unknown, coupled modes also appeared, such as fuel sloshing and roll instability; the use of thinner wings and more slender fuselages meant greater flexibility of the aircraft structure, and the modes associated with this structural flexibility coupled with the rigid-body modes of the aircraft's motion, caused further problems.

One of the first solutions to these problems was the use of a *stability augmentation system* (SAS), which is simply a feedback control system designed to increase the relative damping of a particular mode of the motion of the aircraft. Such an increase in damping is achieved by augmenting one or more of the coefficients of the equations of motion by imposing on the aircraft appropriate forces or moments as a result of actuating the control surfaces in response to feedback signals derived from appropriate motion variables. After SAS, the following AFCS modes were developed: *sideslip suppression SAS*, *pitch attitude hold*, *autothrottle (speed control system)*, *mach hold*, *height hold*, and *turn coordination systems*.

An integrated flight control system is a collection of such AFCS modes in a single comprehensive system, with particular modes being selected by the pilot to suit the task required for any particular phase of flight. In the past such functions were loosely referred to as an *autopilot*, but that name was a trademark registered by the German company Siemens in 1928. Today, AFCS not only augment the stability of an aircraft, but they can follow path and manoeuvre commands, thereby providing the means of automatic tracking and navigation; they can perform automatic take-off and landing; they can provide structural mode control, gust load alleviation, and active ride control.

1.5 BRIEF HISTORY OF FLIGHT CONTROL SYSTEMS

The heavier-than-air machine designed and built by Hiram Maxim in 1891 was colossal for its time: it was 34 m long and weighed 3 600 kg. Even now, the largest propeller to be seen in the aviation collection of the Science Museum in London is one of the pair used by Maxim. It was obvious to Maxim, if to no-one else at the time, that when his aircraft flew, its longitudinal stability would be inadequate, for he installed in the machine a flight control system which used an actuator to deflect the elevator and employed a gyroscope to provide a feedback signal. It was identical, except in inconsequential detail, to a present-day pitch attitude control system. Two of the minor details were the system's weight, over 130 kg, and its power source, steam. The concept remains unique.

Between 1910 and 1912 the American father-and-son team, the Sperrys, developed a two-axis stabilizer system in which the actuators were powered by compressed air and the gyroscopes were also air-driven. The system could maintain both pitch and bank angles simultaneously and, from a photographic record of a celebrated demonstration flight, in which Sperry Snr is seen in the open cockpit, with his arms stretched up above his head, and a mechanic is standing on the upper surface of the upper wing at the starboard wing tip, maintaining level flight automatically was easily within its capacity.

During World War I, aircraft design improved sufficiently to provide, by the sound choice of size, shape and location of the aerodynamic control surfaces, adequate stability for pilots' needs. Many aircraft were still unstable, but not dangerously so, or, to express that properly, the degree of damage was acceptable in terms of the loss rates of pilots and machines.

In the 1920s, however, it was found that, although the early commercial airliners were quite easy to fly, it was difficult to hold heading in poor visibility. Frequently, in such conditions, a pilot and his co-pilot had to divide the flying task between them. The pilot held the course by monitoring both the compass and the turn indicator and by using the rudder; the co-pilot held the speed and the attitude constant by monitoring both the airspeed and the pitch attitude indicator and by controlling the airspeed via the engine throttles and the pitch attitude by using the elevator. From the need to alleviate this workload grew the need to control aircraft automatically.

The most extensive period of development of early flight control systems took place between 1922 and 1937: in Great Britain, at the Royal Aircraft Establishment (RAE) at Farnborough; in Germany, in the industrial firms of Askania and Siemens; and in the USA, in Sperrys and at NACA (National Advisory Committee for Aeronautics – now NASA). Like all other flight control systems up to 1922, the RAE's Mk I system was two-axis, controlling pitch attitude and heading. It was a pneumatic system, but its superior performance over its predecessors and competitors was due to the fact that it had been designed scientifically by applying the methods of dynamic stability analysis which had been developed in Great Britain by some very distinguished applied mathematicians and aerodynamicists (see McRuer *et al.*, 1973; Draper, 1981; Hopkin and Dunn, 1947; McRuer and Graham, 1981; Oppelt, 1976). Such comprehensive theoretical analysis, in association with extensive experimental flight tests and trials carried out by the RAF, led to a clear understanding of which particular motion variables were most effective for use as feedback signals in flight control systems.

In 1927, in Germany, the firm of Askania developed a pneumatic system which controlled heading by means of the aircraft's rudder. It used an air-driven gyroscope, designed and manufactured by Sperrys of the USA. The first unit was flight tested on the Graf Zeppelin-LZ127; the system merits mention only because of its registered trade name, Autopilot. However, the Germans soon decided that as a drive medium, air, which is very compressible, gave inferior performance compared to oil, which was considered to be very nearly incompressible. Thus, in its two-axis 'autopilot' of 1935, the Siemens company successfully used hydraulic actuators and thereby established the trend, still followed today, of using hydraulic oil in preference to air, which in turn was used in preference to Maxim's steam. In 1950, the Bristol Aeroplane Company built a four-engined, turbo-prop transport aircraft which used electric actuators, but it was not copied by other manufacturers. At present, NASA and the USAF are actively pursuing a programme of reasearch designed to lead to 'an all-electric airplane' by 1990.

The reader should not infer from earlier statements that the RAE solved every flight control problem on the basis of having adequate theories. In 1934, the Mk IV system, which was a three-axis pneumatic system, was designed for installation in the Hawker Hart, a biplane in service with the RAF. In flight, a considerable number of stability problems were experienced and these were never solved. However, when the same system was subsequently fitted to the heavy bombers then entering RAF service (the Hampdens, Whitleys and Wellingtons) all the stability problems vanished and no satisfactory reasons for this improvement were ever adduced. (McRuer and Graham (1981) suggest that the increased inertia and the consequently slower response of the heavier aircraft were the major improving factors.)

In 1940, the RAE had developed a new AFCS, the Mk VII, which was again two-axis and pneumatic, but, in the longitudinal axis, used both airspeed and its rate of change as feedback signals, and, in the lateral axis, moved the ailerons in response to a combination of roll and yaw angles. At cruising speed in calm weather the system was adjudged by pilots to give the best automatic control yet devised. But, in some aircraft at low speeds, and in all aircraft in turbulence, the elevator motion caused such violent changes in the pitch attitude that the resultant vertical acceleration so affected the fuel supply that the engines stopped. It was only in 1943 that the problem was eventually solved by Neumark (see Neumark, 1943) who conducted an analysis of the problem entirely by time-domain methods. He used a formulation of the aircraft dynamics that control engineers now refer to as the state equation.

German work did not keep pace with British efforts, since, until very late in World War II, they concentrated on directional and lateral motion AFCSs, only providing a three-axis AFCS in 1944. The American developments had been essentially derived from the Sperry Automatic Pilot used in the Curtiss 'Condors' operated by Eastern Airlines in 1931. Subsequently, electric, three-axis autopilots were developed in the USA by firms such as Bendix, Honeywell and Sperry. The Minneapolis Honeywell C1 was developed from the Norden Stabilized Bomb-sight and was much used in World War II by both the American Air Forces and the Royal Air Force.

The development of automatic landing was due principally to the Blind Landing Experimental Unit of RAE, although in 1943 at the Flight Development Establishment at Rechlin in Germany, at least one aircraft had been landed automatically. The German efforts on flight control at this time were devoted to the systems required for the V1 and V2 missiles. On 23 September 1947 an American Douglas C-54 flew across the Atlantic completely under automatic control, from take-off at Stephenville, in Newfoundland, Canada, to landing at Brize Norton, in England. A considerable effort has been given to developing AFCSs since that time to become the ultra-reliable integrated flight control systems which form the subject of this book. The interested reader is referred to

Hopkin and Dunn (1947), McRuer and Graham (1981), Oppelt (1976) and Howard (1973) for further discussions of the history of flight control systems.

1.7 CONCLUSIONS

In considering the design of an AFCS an engineer will succeed only if he is able both to establish an adequate model representing the appropriate dynamical behaviour of the aircraft to be controlled and to recognize how an effective control system design can be realized.

Consequently, the control engineer working with AFCSs must completely understand the equations of the aircraft's motion, be familiar with their methods of solution, understand the characteristic responses associated with them, know what influence they have on the aircraft's flying qualities, appreciate how atmospheric disturbances can be characterized and know how such disturbances affect performance. Additionally, it is important to understand how primary flying controls can be improved, or their worst effects reduced, so that the match between a human pilot and the aircraft is optimized.

In addition, the theory of control, with its attendant design techniques, must be thoroughly mastered so that it, and they, can be used to produce an AFCS based upon control surface actuators and motion sensors which are available, and whose dynamic behaviour is thoroughly known.

The alternative methods of carrying out the required computation to produce the appropriate control laws have also to be completely understood, and the engineer is expected to be sound in his appreciation of the limitations of whatever particular method was chosen to perform the control design.

Detailed engineering considerations of installing and testing such AFCSs, particularly in regard to certification procedures for airworthiness requirements, and the special reliability considerations of the effect of subsystem failure upon the integrity of the overall system, are special studies beyond this book. The influence of these topics on the final form of the AFCS is profound and represents one of the most difficult aspects of flight control work. Any flight control engineer will be obliged to master both subjects early in his professional career.

1.8 NOTE

1. Sometimes 'centre of mass' and 'centre of gravity' are used interchangeably. For any group of particles in a uniform gravitational field these centres coincide. For spacecraft, their separation is distinctive and this separation results in an appreciable moment due to gravity being exerted on the spacecraft. For aircraft flying in the atmosphere the centres are identically located.

1.9 REFERENCES

DRAPER, C.S. 1981. Control, navigation and guidance. *IEEE Control Systems Magazine*. 1(4): 4–17.

HOPKIN, H.R. and R.W. DUNN. 1947. Theory and development of automatic pilots 1937–1947. RAE report. IAP 1459. August.

HOWARD, R.W. 1973. Automatic flight controls in fixed wing aircraft – the first hundred years. *Aero. J.* 77(11): 553–62.

McRUER, D.T., I.L. ASHKENAS and D.C. GRAHAM. 1973. *Aircraft Dynamics and Automatic Control*. Princeton University Press.

McRUER, D.T. and D.C. GRAHAM. 1981. Eighty years of flight control: triumphs and pitfalls of the systems approach. *J. Guid. and Cont.* 4(4): 353–62.

NEUMARK, S. 1943. The disturbed longitudinal motion of an uncontrolled aeroplane and of an aeroplane with automatic control. ARC R&M 2078. January.

OPPELT, W. 1976. An historical review of Autopilot development, research and theory in Germany. *J. Dyn. Sys., Meas. and Cont.* 98(3): 215–23.

2

The Equations of Motion of an Aircraft

2.1 INTRODUCTION

If the problems associated with designing an AFCS were solely concerned with large area navigation then an appropriate frame of reference, in which to express the equations of motion of an aircraft, would be inertial, with its centre in the fixed stars. But problems involving AFCSs are generally related to events which do not persist: the dynamic situation being considered rarely lasts for more than a few minutes. Consequently, a more convenient inertial reference frame is a tropocentric coordinate system, i.e. one whose origin is regarded as being fixed at the centre of the Earth: the Earth axis system. It is used primarily as a reference system to express gravitational effects, altitude, horizontal distance, and the orientation of the aircraft. A set of axes commonly used with the Earth axis system is shown in Figure 2.1; the axis, X_E, is chosen to point north, the axis, Y_E, then pointing east with the orthogonal triad being completed when the axis, Z_E, points down. If the Earth axis system is used as a basic frame of reference, to which any other axis frames employed in the study are referred, the aircraft itself

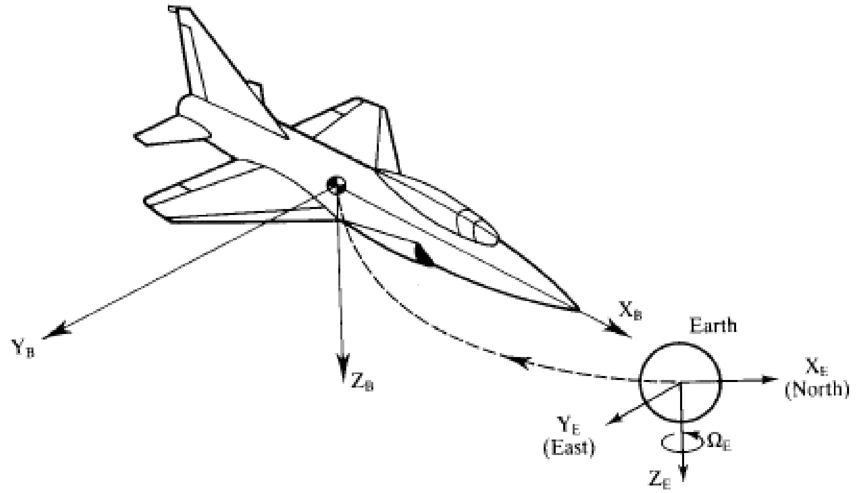


Figure 2.1 Earth axis system.

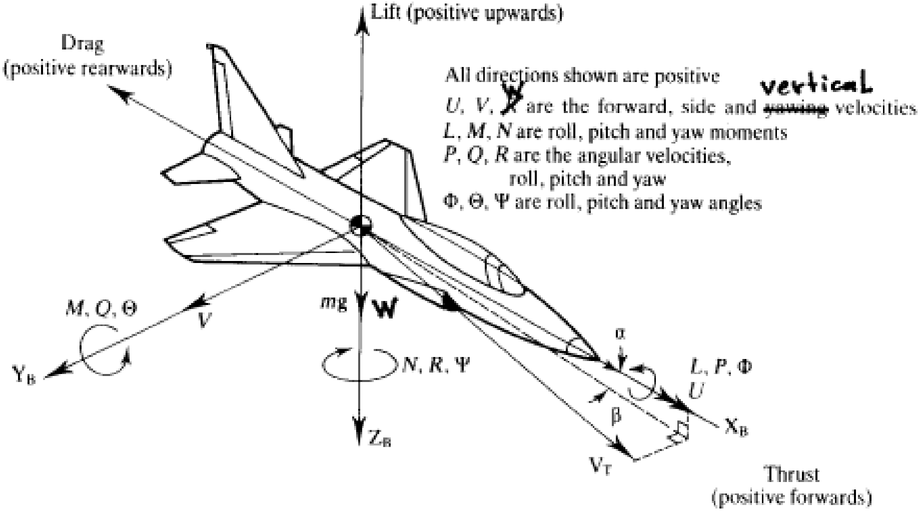


Figure 2.2 Body axis system.

must then have a suitable axis system. Several are available which all find use, to a greater or lesser extent, in AFCS work. The choice of axis system governs the form taken by the equations of motion. However, only body-fixed axis systems, i.e. only systems whose origins are located identically at an aircraft's centre of gravity, are considered in this book. For such a system, the axis, X_B, points forward out of the nose of the aircraft; the axis, Y_B, points out through the starboard (right) wing, and the axis, Z_B, points down (see Figure 2.2). Axes X_B, Y_B and Z_B emphasize that it is a body-fixed axis system which is being used. Forces, moments and velocities are also defined. By using a system of axes fixed in the aircraft the inertia terms, which appear in the equations of motion, may be considered to be constant. Furthermore, the aerodynamic forces and moments depend only upon the angles, α and β, which orient the total velocity vector, V_T, in relation to the axis, X_B. The angular orientation of the body axis system with respect to the Earth axis system depends strictly upon the orientation sequence. This sequence of rotations is customarily taken as follows (see Thelander, 1965):

1. Rotate the Earth axes, X_E, Y_E, and Z_E, through some azimuthal angle, Ψ, about the axis, X_E, to reach some intermediate axes X_1, Y_1 and Z_1.
2. Next, rotate these axes X_1, Y_1 and Z_1 through some angle of elevation, Θ, about the axis Y_1 to reach a second, intermediate set of axes, X_2, Y_2, and Z_2.
3. Finally, the axes X_2, Y_2 and Z_2 are rotated through an angle of bank, Φ, about the axis, X_2, to reach the body axes X_B, Y_B and Z_B.

Three other special axis systems are considered here, because they can be found to have been used sufficiently often in AFCS studies. They are: the stability axis system; the principal axis system; and the wind axis system. In AFCS work, the most commonly used system is the stability axis system.

2.2 AXIS (COORDINATE) SYSTEMS

2.2.1 The Stability Axis System

The axis X_s is chosen to coincide with the velocity vector, $\mathbf{V}_T$, at the start of the motion. Therefore, between the X-axis of the stability axis system and the X-axis of the body axis system, there is a trimmed angle of attack, α_o. The equations of motion derived by using this axis system are a special subset of the set derived by using the body axis system.

2.2.2 The Principal Axis System

This set of body axes is specially chosen to coincide with the principal axes of the aircraft. The convenience of this system resides in the fact that in the equations of motion, all the product of inertia terms are zero, which greatly simplifies the equations.

2.2.3 The Wind Axis System

Because this system is oriented with respect to the aircraft's flight path, time-varying terms which correspond to the moments and cross-products of inertia appear in the equations of motion. Such terms considerably complicate the analysis of aircraft motion and, consequently, wind axes are not used in this text. They have appeared frequently, however, in American papers on the subject.

2.2.4 Sensor Signals

Because an AFCS uses feedback signals from motion sensors, it is important to remember that such signals are relative to the axis system of the sensor and not to the body-fixed axis system of the aircraft. This simple fact can sometimes cause the performance obtained from an AFCS to be modified and, in certain flight tasks, may have to be taken into account. However, in straight and level flight at cruise it is insignificant.

2.3 THE EQUATIONS OF MOTION OF A RIGID BODY AIRCRAFT

2.3.1 Introduction

The treatment given here closely follows that of McRuer *et al.* (1953).

It is assumed, first, that the aircraft is rigid-body; the distance between any points on the aircraft do not change in flight. Special methods to take into account the flexible motion of the airframe are treated in Chapter 4. When the aircraft can be assumed to be a rigid body moving in space, its motion can be considered to have six degrees of freedom. By applying Newton's Second Law to that rigid body the equations of motion can be established in terms of the translational and angular accelerations which occur as a consequence of some forces and moments being applied to the aircraft.

In the introduction to this chapter it was stated that the form of the equations of motion depends upon the choice of axis system, and a few of the advantages of using a body-fixed axis system were indicated there. In the development which follows, a body axis system is used with the change to the stability axis system being made at an appropriate point later in the text. In order to be specific about the atmosphere in which the aircraft is moving, it is also assumed that the inertial frame of reference does not itself accelerate, in other words, the Earth is taken to be fixed in space.

2.3.2 Translational Motion

From Newton's Second Law it can be deduced that:

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}\{m\mathbf{V}_{\mathrm{T}}\} \tag{2.1}$$

$$\mathbf{M} = \frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{H}\} \tag{2.2}$$

where **F** represents the sum of all externally applied forces, **M** represents the sum of all applied torques, and **H** is the angular momentum.

The sum of the external forces has three components: aerodynamic, gravitational and propulsive. In every aircraft some part of the propulsive (thrust) force is produced by expending some of the vehicle's mass. But it can easily be shown[1] that if the mass, m, of an aircraft is assumed to be constant, the thrust, which is a force equal to the relative velocity between the exhausted mass and the aircraft and the change of the aircraft's mass/unit time, can be treated as an external force without impairing the accuracy of the equations of motion. If it is assumed, for the present, that there will be no change in the propulsive force, changes in the aircraft's state of motion from its equilibrium state can occur if and only if there are changes in either the aerodynamic or gravitational forces (or both). If it becomes necessary in a problem to include the changes of thrust (as it will be when dealing with airspeed control systems, for example) only a small extension of the method being outlined here is required. Details in relation to the stability axis system are given in section 2.2. For the present, however, the thrust force can be considered to be contained in the general applied force, **F**.

When carrying out an analysis of an AFCS it is convenient to regard the sums of applied torque and force as consisting of an equilibrium and a perturbational component, namely:

$$\mathbf{F} = \mathbf{F}_0 + \Delta\mathbf{F} = m\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{V}_\mathrm{T}\} \qquad (2.3)[2]$$

$$\mathbf{M} = \mathbf{M}_0 + \Delta\mathbf{M} = \frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{H}\} \qquad (2.4)$$

The subscript 0 denotes the equilibrium component, Δ the component of perturbation. Since the axis system being used as an inertial reference system is the Earth axis system, eqs (2.3) and (2.4) can be re-expressed as:

$$\Delta\mathbf{F} = m\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{V}_\mathrm{T}\}_\mathrm{E} \qquad (2.5)$$

$$\Delta\mathbf{M} = \frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{H}\}_\mathrm{E} \qquad (2.6)$$

By definition, equilibrium flight must be unaccelerated flight along a straight path; during this flight the linear velocity vector relative to fixed space is invariant, and the angular velocity is zero. Thus, both $\mathbf{F}_0$ and $\mathbf{M}_0$ are zero.

The rate of change of $\mathbf{V}_\mathrm{T}$ relative to the Earth axis system is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{V}_\mathrm{T}\}_\mathrm{E} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_\mathrm{T}\Big|_\mathrm{B} + \omega \times \mathbf{V}_\mathrm{T} \qquad (2.7)$$

where ω is the angular velocity of the aircraft with respect to the fixed axis system. Wnen the vectors are expressed in coordinates in relation to the body-fixed axis system, both velocities may be written as the sum of their corresponding components, with respect to X_B, Y_B and Z_B, as follows:

$$\mathbf{V}_\mathrm{T} = \mathbf{i}U + \mathbf{j}V + \mathbf{k}W \qquad (2.8)$$

$$\omega = \mathbf{i}P + \mathbf{j}Q + \mathbf{k}R \qquad (2.9)$$

$$\therefore \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_\mathrm{T}\Big|_\mathrm{B} = \mathbf{i}\dot{U} + \mathbf{j}\dot{V} + \mathbf{k}\dot{W} \qquad (2.10)$$

and the cross-product, $\omega \times \mathbf{V}_\mathrm{T}$, is given by:

$$\omega \times \mathbf{V}_\mathrm{T} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ U & V & W \end{bmatrix}$$

$$= \mathbf{i}(QW - VR) + \mathbf{j}(UR - PW) + \mathbf{k}(PV - UQ) \tag{2.11}$$

In a similar fashion, the components of the perturbation force can be expressed as

$$\Delta\mathbf{F} = \mathbf{i}\Delta F_x + \mathbf{j}\Delta F_y + \mathbf{k}\Delta F_z \tag{2.12}$$

Hence,

$$\Delta\mathbf{F} = m\{\mathbf{i}(\dot{U} + QW - VR) + \mathbf{j}(\dot{V} + UR - PW) + \mathbf{k}(\dot{W} + PV - UQ)\} \tag{2.13}$$

From which it can be inferred that:

$$\Delta F_x = m(\dot{U} + QW - VR) \tag{2.14}$$

$$\Delta F_y = m(\dot{V} + UR - PW) \tag{2.15}$$

$$\Delta F_z = m(\dot{W} + VP - UQ) \tag{2.16}$$

Rather than continue the development using the cumbersome notation, ΔF_i, to denote the ith component of the perturbational force, it is proposed to follow the American custom and use the following notation:

$$\Delta X \triangleq \Delta F_x \qquad \Delta Y \triangleq \Delta F_y \qquad \Delta Z \triangleq \Delta F_z \tag{2.17}$$

It must be remembered that now X, Y and Z denote *forces*. With these substitutions in eqs (2.14)–(2.16), the equations of translational motion can be expressed as:

$$\Delta X = m(\dot{U} + QW - VR) \tag{2.18}$$

$$\Delta Y = m(\dot{V} + UR - PW) \tag{2.19}$$

$$\Delta Z = m(\dot{W} + VP - UQ) \tag{2.20}$$

2.3.3 Rotational Motion

For a rigid body, angular momentum may be defined as:

$$\mathbf{H} = I\omega \tag{2.21}$$

The inertia matrix, I, is defined as:

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \tag{2.22}$$

where I_{ii} denotes a moment of inertia, and I_{ij} a product of inertia $j \neq i$.

$$\mathbf{M} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H} + \boldsymbol{\omega} \times \mathbf{H} \tag{2.23}$$

Transforming from body axes to the Earth axis system (see Gaines and Hoffman, 1972) allows eq. (2.23) to be re-expressed as:

$$\mathbf{M} = I \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\omega} \right\} + \boldsymbol{\omega} \times \mathbf{H} \tag{2.24}$$

However,

$$\boldsymbol{\omega} \times \boldsymbol{\omega} \triangleq 0 \tag{2.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\omega} = \mathbf{i}\dot{P} + \mathbf{j}\dot{Q} + \mathbf{k}\dot{R} \tag{2.26}$$

and

$$\boldsymbol{\omega} \times \mathbf{H} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ h_x & h_y & h_z \end{bmatrix} \tag{2.27}$$

where h_x, h_y and h_z are the components of $\mathbf{H}$ obtained from expanding eq. (2.21) thus:

$$h_x = I_{xx}P - I_{xy}Q - I_{xz}R \tag{2.28}$$

$$h_y = -I_{yx}P + I_{yy}Q - I_{yz}R \tag{2.29}$$

$$h_z = -I_{zx}P - I_{zy}Q + I_{zz}R \tag{2.30}$$

In general, aircraft are symmetrical about the plane XZ, and consequently it is generally the case that:

$$I_{xy} = I_{yz} = 0 \tag{2.31}$$

Therefore:

$$h_x = I_{xx}P - I_{xz}R \tag{2.32}$$

$$h_y = I_{yy}Q \tag{2.33}$$

$$h_z = -I_{zx}P + I_{zz}R \tag{2.34}$$

and

$$\Delta M_x = I_{xx}\dot{P} - I_{xz}(\dot{R} + PQ) + QR(\mathrm{I}_{zz} - \mathrm{I}_{yy}) \tag{2.35}$$

$$\Delta M_y = I_{yy}\dot{Q} + I_{xz}(P^2 - R^2) + PR(I_{xx} - I_{zz}) \tag{2.36}$$

$$\Delta M_z = I_{zz}\dot{R} - I_{xz}\dot{P} + PQ(I_{yy} - I_{xx}) + I_{zz}QR \tag{2.37}$$

Again, following American usage:

$$\Delta M_x = \Delta L \qquad \Delta M_y = \Delta M \qquad \Delta M_z = \Delta N \tag{2.38}$$

where L, M and N are moments about the rolling, pitching and yawing axes respectively.

$$\Delta L = I_{xx}\dot{P} - I_{xz}(\dot{R} + PQ) + (I_{zz} - I_{yy})QR \tag{2.39}$$

$$\Delta M = I_{yy}\dot{Q} + I_{xz}(P^2 - R^2) + (I_{xx} - I_{zz})PR \tag{2.40}$$

$$\Delta N = I_{zz}\dot{R} - I_{xz}\dot{P} + PQ(I_{yy} - I_{xx}) + I_{xz}QR \tag{2.41}$$

2.3.4 Some Points Arising from the Derivation of the Equations

It is worth emphasizing here that the form of equations arrived at, having used a body axis system, is not entirely convenient for flight simulation work (Fogarty and Howe, 1969). For example, suppose a fighter aircraft has a maximum velocity of 600 m s^{-1} and a maximum angular velocity Q_B of 2.0 rad s^{-1}. The term, UQ, in eq. (2.20) can have a value as large as 1 200 m s^{-2}, i.e. 120 g, whereas the term, ΔZ, the normal acceleration due to the external forces (primarily aerodynamic and gravitational) may have a maximum value in the range 10.0 to 20.0 m s^{-2} (i.e. 1–2 g). It can be seen, therefore, how a (dynamic) acceleration of very large value, perhaps fifty times greater than the physical accelerations, can occur in the equations merely as a result of the high rate of rotation experienced by the body axis system. Furthermore, it can be seen from inspection of eqs (2.18)–(2.20) how angular motion has been coupled into translational motion. Moreover, on the right-hand side of eqs (2.39)–(2.41) the third term is a non-linear, inertial coupling term. For large aircraft, such as transports, which cannot generate large angular rates, these terms are frequently neglected so that the moment equations become:

$$\Delta L = I_{xx}\dot{P} - I_{xz}(\dot{R} + PQ) \tag{2.42}$$

$$\Delta M = I_{yy}\dot{Q} + I_{xz}(P^2 - R^2) \tag{2.43}$$

$$\Delta N = I_{zz}\dot{R} - I_{xz}(\dot{P} - QR) \tag{2.44}$$

A number of other assumptions are frequently invoked in relation to these equations:

1. Sometimes, for a particular aircraft, the product of inertia, I_{xz}, is sufficiently small to allow of its being neglected. This often happens when the body axes, X_B, Y_B, and Z_B have been chosen to almost coincide with the principal axes.
2. For aircraft whose maximum values of angular velocity are low, the terms PQ, QR, and $P^2 - R^2$ can be neglected.
3. Since R^2 is frequently very much smaller than P^2, it is often neglected.

It is emphasized, however, that the neglect of such terms can only be practised after very careful consideration of both the aircraft's characteristics and the AFCS problem being considered. Modern fighter aircraft, for example, may lose control as a result of roll/pitch inertial coupling. In such aircraft, pitch-up is sensed when a roll manoeuvre is being carried out. When an AFCS is fitted, such a sensor signal would cause an elevator deflection to be commanded to provide a nose-down attitude until the elevator can be deflected no further and the aircraft cannot be controlled. Such a situation can happen whenever the term $(I_{xx} - I_{zz})PR$ is large enough to cause an uncontrollable pitching movement.

2.3.5 Contributions to the Equations of Motion of the Forces Due to Gravity

The forces due to gravity are always present in an aircraft; however, by neglecting any consideration of gradients in the gravity field, which are important only in extra-atmospheric flight if all other external forces are essentially non-existent, it can be properly assumed that gravity acts at the centre of gravity (c.g.) of the aircraft. Hence, since the centres of mass and gravity coincide in an aircraft, there is no external moment produced by gravity about the c.g. Hence, for the body axis system, gravity contributes only to the external force vector, **F**.

The gravitational force acting upon an aircraft is most obviously expressed in terms of the Earth axes. With respect to these axes the gravity vector, $m\mathbf{g}$, is directed along the Z_E axis. Figure 2.3 shows the alignment of the gravity vector with respect to the body-fixed axes. In Figure 2.3 Θ represents the angle between the gravity vector and the Y_BZ_B plane; the angle is positive when the nose of the aircraft goes up. Φ represents the bank angle between the axis Z_B and the projection of the gravity vector on the Y_BZ_B plane; the angle is positive when the right wing is down. Direct resolution of the vector $m\mathbf{g}$, into X, Y and Z components produces:

$$\begin{aligned}
\delta X &= m\mathbf{g}\sin[-\Theta] = -m\mathbf{g}\sin\Theta \\
\delta Y &= m\mathbf{g}\cos[-\Theta]\sin\Phi = m\mathbf{g}\cos\Theta\sin\Phi \qquad (2.45) \\
\delta Z &= m\mathbf{g}\cos[-\Theta]\cos\Phi = m\mathbf{g}\cos\Theta\cos\Phi
\end{aligned}$$

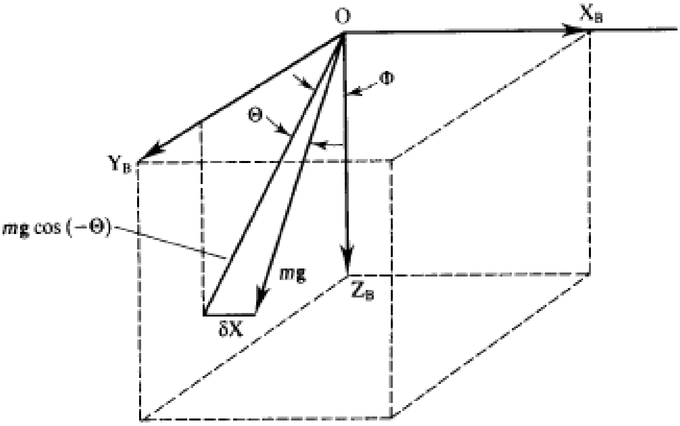


Figure 2.3 Orientation of gravity vector with body axis systems.

In general, the angles Θ and Φ are not simply the integrals of the angular velocity P and Q; in effect, two new motion variables have been introduced and it is necessary to relate them and their derivatives to the angular velocities, P, Q and R. How this is done depends upon whether the gravitational vertical seen from the aircraft is fixed or whether it rotates relative to inertial space. Aircraft speeds being very low compared to orbital velocities, the vertical may be regarded as fixed. In very high speed flight the vertical will be seen as rotating and the treatment which is being presented here will then require some minor amendments.

The manner in which the angular orientation and velocity of the body axis system with respect to the gravity vector is expressed depends upon the angular velocity of the body axes about the vector $m\mathbf{g}$. This angular velocity is the azimuth rate, $\dot{\Psi}$; it is not normal to either $\dot{\Phi}$ or $\dot{\Theta}$, but its projection in the Y_BZ_B plane is normal to both (see Figure 2.4). By resolution, it is seen that:

$$\begin{aligned} P &= \dot{\Phi} - \dot{\Psi} \sin\Theta \\ Q &= \dot{\Theta}\cos\Phi + \dot{\Psi}\cos\Theta\sin\Phi \\ R &= -\dot{\Theta}\sin\Phi + \dot{\Psi}\cos\Theta\cos\Phi \end{aligned} \tag{2.46}$$

Also,

$$\begin{aligned} \dot{\Phi} &= P + \dot{\Psi}\sin\Theta \\ \dot{\Theta} &= Q\cos\Phi - R\sin\Phi \\ \dot{\Psi} &= \frac{R\cos\Phi}{\cos\Theta} + \frac{Q\sin\Phi}{\cos\Theta} \end{aligned} \tag{2.47}$$

Using substitution, it is easy to show that:

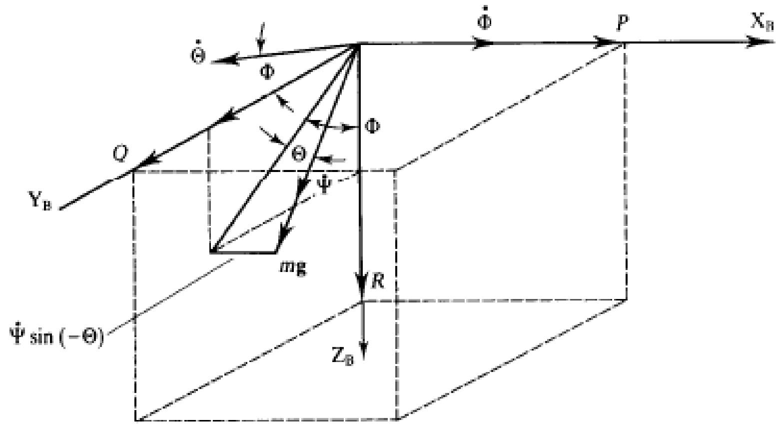


Figure 2.4 Angular orientation and velocities of gravity vector, **g**, relative to body axis.

$$\dot{\Phi} = P + R \tan\Theta \cos\Phi + Q \tan\Theta \sin\Phi \tag{2.48}$$

Φ, Θ and Ψ are referred to as the Euler angles.

2.3.6 Axis Transformations

The physical relationships established so far depend upon two frames of reference: the Earth axis system and the body axis system. To orient these systems one to another requires the use of axis transformations. Any set of axes can be obtained from any other set by a sequence of three rotations. For each rotation a transformation matrix is applied to the variables. The total transformation array is obtained simply by taking the product of the three matrices, multiplied in the order of the rotations. In aircraft dynamics, the most common set of transformations is that between the Earth axis system which incorporates the gravity vector, **g**, as one axis, and the body-fixed axes, X_B, Y_B and Z_B. The rotations follow the usual order: azimuth Ψ, pitch Θ, and roll Φ. The corresponding matrices are:

$$T_\Psi = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.49}$$

$$T_\Theta = \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix} \tag{2.50}$$

$$T_\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \tag{2.51}$$

The complete transformation matrix T is called the direction cosine array and is defined as:

$$T = [T_\Psi][T_\Theta][T_\Phi] \quad [T_\phi][T_\Theta][T_\psi] \tag{2.52}$$

Before expressing the matrix T in full, a notational shorthand is proposed whereby a term such as $\cos\xi$ is written as $c\xi$ and a term such as $\sin\xi$ is written as $s\xi$. Thus:

$$T = \begin{bmatrix} c\Psi c\Theta & s\Psi s\Theta \;\; (s\Psi c\Theta) & -s\Theta \\ (c\Psi s\Theta s\Phi - s\Psi c\Phi) & (s\Psi s\Theta s\Phi + c\Psi c\Phi) & c\Theta s\Phi \\ (c\Psi s\Theta c\Phi + s\Psi s\Phi) & (s\Psi s\Theta c\Phi - c\Psi s\Phi) & c\Theta c\Phi \end{bmatrix} \tag{2.53}$$

It is worth noting that the order of rotation $\Psi - \Theta - \Phi$ is that which results in the least complicated resolution of the gravity vector **g** into the body axis system. It can easily be shown that:

$$\mathbf{g} = g\{- s\Theta\mathbf{i} + c\Theta s\Phi\mathbf{j} + c\Theta c\Phi\mathbf{k}\} \tag{2.54}$$

Another practical advantage is that the angles are those which are measured by a typically oriented vertical gyroscope. A two degree of freedom, gravity erected, vertical gyroscope, oriented such that the bearing axis of its outer gimbal lies along OX_B, measures on its inner and outer gimbals the Euler angles Θ and Φ, respectively.

2.3.7 Linearization of the Inertial and Gravitational Terms

Equations (2.14)–(2.16) and (2.39)–(2.41) represent the inertial forces acting on the aircraft. Equation (2.45) represents the contribution of the forces due to gravity to those equations. All these forces are proportional to the mass of the aircraft. Consequently, these terms may be conveniently combined into components to represent the accelerations which would be measured by sensors located on the aircraft in such a manner that the input axes of the sensors would be coincident with the body axes X_B, Y_B and Z_B. The external forces acting on the aircraft can be re-expressed as:

$$\begin{aligned} X &= \Delta X + \delta X \\ Y &= \Delta Y + \delta Y \\ Z &= \Delta Z + \delta Z \end{aligned} \tag{2.55}$$

where δX, δY and δZ are the gravitational terms and ΔX, ΔY and ΔZ represent the aerodynamic and thrust forces. For notational convenience, ΔL, ΔM and ΔN are now denoted by L, M and N. Thus the equations of motion of the rigid body, for its six degrees of freedom, may be expressed as:

$$\begin{aligned} X - ma_{x_{cg}} &= m[\dot{U} + QW - RV + g\sin\Theta] \\ Y - ma_{y_{cg}} &= m[\dot{V} + RU - PW - g\cos\Theta\sin\Phi] \\ Z - ma_{z_{cg}} &= m[\dot{W} + PV - QU - g\cos\Theta\cos\Phi] \\ L &= \dot{P}I_{xx} - I_{xz}(\dot{R} + PQ) + (I_{zz} - I_{yy})QR \\ M &= \dot{Q}I_{yy} + I_{xz}(P^2 - R^2) + (I_{xx} - I_{zz})PR \\ N &= \dot{R}I_{zz} - I_{xz}\dot{P} + PQ(I_{yy} - I_{xx}) + I_{xz}QR \end{aligned} \tag{2.56}$$

The auxiliary equations of eq. (2.46) must also be used since they relate Ψ, Θ and Φ to R, Q and P.

The equations which constitute eq. (2.56) are non-linear since they contain terms which comprise the product of dependent variables, the squares of dependent variables, and some of the terms are transcendental. Solutions of such equations cannot be obtained analytically and would require the use of a computer. Some simplification is possible, however, by considering the aircraft to comprise two components: a mean motion which represents the equilibrium, or trim, conditions, and a dynamic motion which accounts for the perturbations about the mean motion. In this form of analysis it is customary to assume that the perturbations are small. Thus, every motion variable is considered to have two components. For example:

$$
\begin{aligned}
U &\triangleq U_0 + u \qquad & R &\triangleq R_0 + r \\
Q &\triangleq Q_0 + q \qquad & M &\triangleq M_0 + m_I \quad \text{etc.}
\end{aligned}
\tag{2.57}
$$

The trim, or equilibrium, values are denoted by a subscript 0 and the small perturbation values of a variable are denoted by the lower case letter.[3]

In trim there can be no translational or rotational acceleration. Hence, the equations which represent the trim conditions can be expressed as:

$$
\begin{aligned}
X_0 &= m[Q_0W_0 - R_0V_0 + g\sin\Theta_0] \\
Y_0 &= m[U_0R_0 - P_0W_0 - g\cos\Theta_0\sin\Phi_0] \\
Z_0 &= m[P_0V_0 - Q_0U_0 - g\cos\Theta_0\cos\Phi_0] \\
L_0 &= Q_0R_0(I_{zz} - I_{yy}) - P_0Q_0I_{xz} \\
M_0 &= (P_0^2 - R_0^2)I_{xz} + (I_{xx} - I_{zz})P_0R_0 \\
N_0 &= I_{xz}Q_0R_0 + (I_{yy} - I_{xx})P_0Q_0
\end{aligned}
\tag{2.58}
$$

Steady rolling, pitching and yawing motion can occur in the trim condition; the equations which define P_0, Q_0 and R_0 are given by eq. (2.46) but with Φ, Θ and Ψ being subscripted by 0.

The perturbed motion can be found either by substituting eq. (2.57) into (2.56), expanding the terms and then subtracting eq. (2.58) from the result, or by differentiating both sides of eq. (2.56). When perturbations from the mean conditions are small, the sines and cosines can be approximated to the angles themselves and the value unity, respectively. Moreover, the products and squares of the perturbed quantities are negligible. Thus, the perturbed equations of motion for an aircraft can be written as:

$$
\begin{aligned}
dX &= m[\dot{u} + W_0q + Q_0w - V_0r - R_0v + g\cos\Theta_0\theta] \\
dY &= m[\dot{v} + U_0r + R_0u - W_0p - P_0w - (g\cos\Theta_0\cos\Phi_0)\phi \\
&\qquad + (g\sin\Theta_0\sin\Phi_0)\theta] \\
dZ &= m[\dot{w} + V_0p + P_0v - U_0q - Q_0u + (g\cos\Theta_0\sin\Phi_0)\phi \\
&\qquad + (g\sin\Theta_0\cos\Phi_0)\theta]
\end{aligned}
\tag{2.59}
$$

$$dL = I_{xx}\dot{p} - I_{xz}\dot{r} + (I_{zz} - I_{yy})(Q_0 r + R_0 q) - I_{xz}(P_0 q + Q_0 p)$$

$$dM = I_{yy}\dot{q} + (I_{xx} - I_{zz})(P_0 r + R_0 p) - (2R_0 r - 2P_0 p)I_{xz}$$

$$dN = I_{zz}\dot{r} - I_{xz}\dot{p} + (I_{yy} - I_{xx})(P_0 q + Q_0 p) + I_{xz}(Q_0 r + R_0 q)$$

see bottom ↓

where Ψ_0, Θ_0 and Φ_0 have been used to represent steady orientations, and Ψ, θ and ϕ the perturbations in the Euler angles. Equations (2.59) are now linear. Obviously, perturbation equations are required for the auxiliary set of equations given as eq. (2.46), because the gravitional forces must be perturbed by any small change in the orientation of the body axis system with respect to the Earth axis system. However, the full set of perturbed, auxiliary equations is rarely used since it is complicated. But the components of angular velocity which represent the rotation of the body-fixed axes X_B, Y_B and Z_B relative to the Earth axes X_E, Y_E and Z_E are sometimes required. These are:

$$\begin{aligned} p &= \dot{\phi} - \dot{\Psi}\sin\Theta_0 - \theta(\dot{\Psi}_0\cos\Theta_0) \\ q &= \dot{\phi}\cos\Phi_0 - \theta(\dot{\Psi}_0\sin\Phi\sin\Theta_0) + \dot{\Psi}\sin\Psi_0\cos\Theta \\ &\quad + \phi(\dot{\Psi}_0\cos\Theta_0\cos\Phi_0 - \dot{\Theta}_0\sin\Phi_0) \\ r &= \dot{\Psi}\cos\Theta_0\cos\Phi_0 - \phi(\dot{\Psi}_0\cos\Theta_0\sin\Phi_0 + \dot{\Psi}_0\cos\Phi_0) \\ &\quad - \dot{\theta}\sin\Phi_0 - \theta(\dot{\Psi}_0\sin\Theta_0\cos\Phi_0) \end{aligned} \tag{2.60}$$

Although these equations are linear, they are still too cumbersome for general use owing to the completely general trim conditions which have been allowed. What is commonly done in AFCS studies is to consider flight cases with simpler trim conditions, a case of great interest being, for example, when an aircraft has been trimmed to fly straight in steady, symmetric flight, with its wings level. Steady flight is motion with the rates of change of the components of linear and angular velocity being zero. Possible steady flight conditions include level turns, steady sideslip and helical turns. Steady pitching flight must be regarded as merely a 'quasi-steady' condition because $\dot{U}$ and $\dot{W}$ cannot both be zero for any appreciable time if Q is not zero. Straight flight is motion with the components of angular velocity being zero. Steady sideslips and dives and climbs without longitudinal acceleration are straight flight conditions. Symmetric flight is motion in which the plane of symmetry of the aircraft remains fixed in space throughout the manoeuvre taking place. Dives and climbs with wings level, and pull-ups without sideslipping, are examples of symmetric flight. Sideslip, rolls and turns are typical asymmetric flight conditions. The significance of the specified trim conditions may be judged when the following implications are understood:

1. That straight flight implies $\dot{\Psi}_0 = \dot{\Theta}_0 = 0$.
2. That symmetric flight implies $\Psi_0 = V_0 = 0$.
3. That flying with wings level implies $\Phi_0 = 0$.

For this particular trimmed flight state, the aircraft will have particular values of

ψ should be Lower case. This problem goes on over the next pages until it gets solved on page 37.

U_0, W_0 and Θ_0. These may be zero, but for conventional aircraft the steady forward speed, U_0, must be greater than the stall speed if flight is to be sustained. However, certain rotary wing and V/STOL aircraft can achieve a flying state in which U_0, W_0 and Θ_0 may be zero; when U_0 and W_0 are simultaneously zero the aircraft is said to be hovering.

Hence, for straight, symmetric flight with wings level, the equations which represent translational motion in eq. (2.59) become:

$$\begin{aligned} x &= m[\dot{u} + W_0 q + Q_0 w - R_0 v + g \cos\Theta_0 \theta] \\ y &= m[\dot{v} + U_0 r + R_0 u - W_0 p - P_0 w - g \cos\Theta_0 \phi] \\ z &= m[\dot{w} + P_0 v - U_0 q - Q_0 u + g \sin\Theta_0 \theta] \end{aligned} \tag{2.61}$$

The equations (2.59) which represent rotational motion are unaffected. Equation (2.60), however, becomes:

$$\begin{aligned} p &= \dot{\phi} - \dot{\Psi} \sin\Theta_0 \\ q &= \dot{\theta} \\ r &= \dot{\Psi} \cos\Theta_0 \end{aligned} \tag{2.62}$$

From the same expression, for this trimmed flight state, it may be assumed that:

$$Q_0 = P_0 = R_0 = 0 \tag{2.63}$$

Therefore, it is possible to write eqs (2.59) and (2.61) in the new form:

$$\begin{aligned} x &= m[\dot{u} + W_0 q + g \cos\Theta_0 \theta] \\ y &= m[\dot{v} + U_0 r - W_0 p - g \cos\Theta_0 \phi] \\ z &= m[\dot{w} - U_0 q + g \sin\Theta_0 \theta] \\ l &= I_{xx}\dot{p} - I_{xz}\dot{r} \\ m_1 &= I_{yy}\dot{q} \\ n &= I_{zz}\dot{r} - I_{xz}\dot{p} \end{aligned} \tag{2.64}$$

Consideration of eq. (2.64) indicates not only that the equations have been simplified, but that the set can be separated into two distinct groups which are given below:

$$\begin{aligned} x &= m[\dot{u} + W_0 q + g \cos\Theta_0 \theta] \\ z &= m[\dot{w} - U_0 q + g \sin\Theta_0 \theta] \\ m_1 &= I_{yy}\dot{q} \end{aligned} \tag{2.65}$$

and

$$\begin{aligned} y &= m[\dot{v} + U_0 r - W_0 p - g \cos\Theta_0 \phi] \\ l &= I_{xx}\dot{p} - I_{xz}\dot{r} \end{aligned} \tag{2.66}$$

$$n = I_{zz}\dot{r} - I_{xz}\dot{p}$$

In eq. (2.65) the dependent variables are u, w, q and θ and these are confined to the plane X_BZ_B. The set of equations is said to represent the longitudinal motion. The lateral/directional motion, consisting of sideslip, rolling and yawing motion is represented in eq. (2.66). Although it appears from this equation that the sideslip is not coupled to the rolling and yawing accelerations, the motion is, however, coupled (at least implicitly). In practice, a considerable amount of coupling can exist as a result of aerodynamic forces which are contained within the terms on the left-hand side of the equations.

It is noteworthy that this separation of lateral and longitudinal equations is merely a separation of gravitational and inertial forces: this separation is possible only because of the assumed trim conditions. But 'in flight', the six degrees of freedom model may be coupled strongly by those forces and moments which are associated with propulsion or with the aerodynamics.

2.4 COMPLETE LINEARIZED EQUATIONS OF MOTION

2.4.1 Expansion of Aerodynamic Force and Moment Terms

To expand the left-hand side of the equations of motion, a Taylor series is used about the trimmed flight condition. Thus, for example,

$$z = \frac{\partial Z}{\partial u}u + \frac{\partial Z}{\partial \dot{u}}\dot{u} + \frac{\partial Z}{\partial w}w + \frac{\partial Z}{\partial \dot{w}}\dot{w} + \frac{\partial Z}{\partial q}q + \frac{\partial Z}{\partial \dot{q}}\dot{q} + \frac{\partial Z}{\partial \delta_E}\delta_E + \frac{\partial Z}{\partial \dot{\delta}_E}\dot{\delta}_E + \ldots \tag{2.67}$$

Equation (2.67) supposes that the perturbed force z has a contribution from only one control surface, the elevator. However, if any other control surface on the aircraft being considered were involved, additional terms, accounting for their contribution to z, would be used. For example, if changes of thrust (T), and the deflection of flaps (F) and symmetrical spoilers (sp) were also used as controls for longitudinal motion, additional terms, such as

$$\frac{\partial Z}{\partial \delta_T}\delta_T, \quad \frac{\partial Z}{\partial \delta_F}\delta_F \text{ and } \frac{\partial Z}{\partial \delta_{sp}}\delta_{sp}$$

would be added to eq. (2.67). Furthermore, some terms depending on other motion variables, such as θ, are omitted because they are generally insignificant.

For the moment only longitudinal motion is treated, and, for simplicity, it is assumed that only elevator deflection is involved in the control of the aircraft's longitudinal motion. Thus, it is now possible to write eq. (2.65) as:

$$\frac{\partial X}{\partial u}u + \frac{\partial X}{\partial \dot{u}}\dot{u} + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial \dot{w}}\dot{w} + \frac{\partial x}{\partial q}q + \frac{\partial X}{\partial \dot{q}}\dot{q} + \frac{\partial X}{\partial \delta_E}\delta_E$$
$$+ \frac{\partial X}{\partial \dot{\delta}_E}\dot{\delta}_E = m[\dot{u} + W_0 q + g\cos\Theta_0\theta]$$

$$\frac{\partial Z}{\partial u}u + \frac{\partial Z}{\partial \dot{u}}\dot{u} + \frac{\partial Z}{\partial w}w + \frac{\partial Z}{\partial \dot{w}}\dot{w} + \frac{\partial Z}{\partial q}q + \frac{\partial Z}{\partial \dot{q}}\dot{q} + \frac{\partial Z}{\partial \delta_E}\delta_E$$
$$+ \frac{\partial Z}{\partial \dot{\delta}_E}\dot{\delta}_E = m[\dot{w} - U_0 q + g\sin\Theta_0\theta] \qquad (2.68)$$

$$\frac{\partial M}{\partial u}u + \frac{\partial M}{\partial \dot{u}}\dot{u} + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial \dot{w}}\dot{w} + \frac{\partial M}{\partial q}q + \frac{\partial M}{\partial \dot{q}}\dot{q} + \frac{\partial M}{\partial \delta_E}\delta_E$$
$$+ \frac{\partial M}{\partial \dot{\delta}_E}\dot{\delta}_E = I_{yy}\dot{q}$$

To simplify the notation it is customary to make the following substitutions:

$$X_x = \frac{1}{m}\frac{\partial X}{\partial x}$$
$$Z_x = \frac{1}{m}\frac{\partial Z}{\partial x} \qquad (2.69)$$
$$M_x = \frac{1}{I_{yy}}\frac{\partial M}{\partial x}$$

When this substitution is made the coefficients, such as M_x, Z_x, and X_x, are referred to as the stability derivatives.

2.4.2 Equations of Longitudinal Motion

Equation (2.68) may now be rewritten in the following form:

$$\dot{u} = X_u u + X_{\dot{u}}\dot{u} + X_w w + X_{\dot{w}}\dot{w} + X_q q + X_{\dot{q}}\dot{q} - W_0 q$$
$$- g\cos\Theta_0\theta + X_{\delta_E}\delta_E + X_{\dot{\delta}_E}\dot{\delta}_E$$
$$\dot{w} = Z_u u + Z_{\dot{u}}\dot{u} + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q + Z_{\dot{q}}\dot{q} + U_0 q$$
$$- g\sin\Theta_0\theta + Z_{\delta_E}\delta_E + Z_{\dot{\delta}_E}\dot{\delta}_E \qquad (2.70)$$
$$\dot{q} = M_u u + M_{\dot{u}}\dot{u} + M_w w + M_{\dot{w}}\dot{w} + M_q q + M_{\dot{q}}\dot{q}$$
$$+ M_{\delta_E}\delta_E + M_{\dot{\delta}_E}\dot{\delta}_E$$

For completeness, the second equation of (2.62) is usually added to eq. (2.70), i.e.

$$\dot{\theta} = q \qquad (2.70a)$$

From studying the aerodynamic data of a large number of aircraft it becomes evident that not every stability derivative is significant and, frequently, a number can be neglected. However, it is essential to remember that such stability derivatives depend both upon the aircraft being considered and the flight condition which applies. Thus, before ignoring stability derivatives, it is important to check the appropriate aerodynamic data. Without loss of generality it can be assumed that the following stability derivatives are often insignificant, and may be ignored:

$$X_{\dot{u}},\ X_q,\ X_{\dot{w}},\ X_{\dot{\delta}_E},\ Z_{\dot{u}},\ Z_{\dot{w}},\ M_{\dot{u}},\ Z_{\dot{\delta}_E} \text{ and } M_{\dot{\delta}_E}.$$

The stability derivative Z_q is usually quite large but often ignored if the trimmed forward speed, U_0, is large. If the case being studied is hovering motion, then Z_q ought not to be ignored. With these assumptions, the equations of perturbed longitudinal motion, for straight, symmetric flight, with wings level, can be expressed as:

$$\begin{aligned}
\dot{u} &= X_u u + X_w w - W_0 q - g \cos \Theta_0 \theta \\
\dot{w} &= Z_u u + Z_w w + U_0 q - g \sin \Theta_0 \theta + Z_{\delta_E} \delta_E \\
\dot{q} &= M_u u + M_w w + M_{\dot{w}} \dot{w} + M_q q + M_{\delta_E} \delta_E \\
\dot{\theta} &= q
\end{aligned} \tag{2.71}$$

Notice that each term in the first three equations of (2.71) is an acceleration term, but since the motion and control variables, u, w, q, θ and δ_E, have such units as m s^{-1}, and s^{-1} the stability derivatives appearing in these equations are dimensional. It is possible to write similar equations using non-dimensional stability derivatives, and this is frequently done in American literature and is always done in the British system; but when it is done, the resulting equations must be written in terms of 'dimensionless' time. The responses obtained from those equations are then expressed in units of time which differ from real time. If the reader requires details of the use of non-dimensional stability derivatives, Babister (1961) should be consulted. It has been decided in this book to use the form of equations given in (2.71) where dimensional stability derivatives must be used (these are the stability derivatives which are usually quoted in American works) but where time is real. Such a decision makes the design of AFCSs much easier and more direct for it allows direct simulation, and also makes the interpretation of the aircraft responses in terms of flying qualities more straightforward.

2.4.3 Equations of Lateral Motion

From eqs (2.64) and (2.62) the following set of equations applies to lateral motion:

$$\begin{aligned}
y &= m[\dot{v} + U_0 r - W_0 p - g \cos \Theta_0 \phi] \\
l &= I_{xx} \dot{p} - I_{xz} \dot{r} \\
n &= I_{zz} \dot{r} - I_{xz} \dot{p}
\end{aligned} \tag{2.72}$$

$$p = \dot{\phi} - \dot{\Psi} \sin \Theta_0$$
$$r = \dot{\Psi} \cos \Theta_0$$

Expanding the left-hand side of the first three equations results in the following (subscripts A and R indicate aileron and rudder, respectively):

$$\frac{\partial Y}{\partial v} v + \frac{\partial Y}{\partial \dot{v}} \dot{v} + \frac{\partial Y}{\partial r} r + \frac{\partial Y}{\partial \dot{r}} \dot{r} + \frac{\partial Y}{\partial p} p + \frac{\partial Y}{\partial \dot{p}} \dot{p} + \frac{\partial Y}{\partial \delta_A} \delta_A + \frac{\partial Y}{\partial \delta_R} \delta_R$$
$$= m[\dot{v} + U_0 r - W_0 p - g \cos \Theta_0 \phi]$$
$$\frac{\partial L}{\partial v} v + \frac{\partial L}{\partial \dot{v}} \dot{v} + \frac{\partial L}{\partial r} r + \frac{\partial L}{\partial \dot{r}} \dot{r} + \frac{\partial L}{\partial p} p + \frac{\partial L}{\partial \dot{p}} \dot{p} + \frac{\partial L}{\partial \delta_A} \delta_A + \frac{\partial L}{\partial \delta_R} \delta_R$$
$$= I_{xx} \dot{p} - I_{xz} \dot{r} \tag{2.73}$$
$$\frac{\partial N}{\partial v} v + \frac{\partial N}{\partial \dot{v}} \dot{v} + \frac{\partial N}{\partial r} r + \frac{\partial N}{\partial \dot{r}} \dot{r} + \frac{\partial N}{\partial p} p + \frac{\partial N}{\partial \dot{p}} \dot{p} + \frac{\partial N}{\partial \delta_A} \delta_A + \frac{\partial N}{\partial \delta_R} \delta_R$$
$$= I_{zz} \dot{r} - I_{xz} \dot{p}$$

Adopting the more convenient notation, namely:

$$Y_j \triangleq \frac{1}{m} \frac{\partial Y}{\partial j} \qquad L_j \triangleq \frac{1}{I_{xx}} \frac{\partial L}{\partial j} \qquad N_j \triangleq \frac{1}{I_{zz}} \frac{\partial N}{\partial j} \tag{2.74}$$

allows the eqs (2.73) to be written more simply as:

$$\dot{v} = Y_v v + Y_{\dot{v}} \dot{v} + Y_r r + Y_p p + Y_{\dot{r}} \dot{r} + Y_{\dot{p}} \dot{p} + Y_{\delta_A} \delta_A + Y_{\delta_R} \delta_R - U_0 r + W_0 p + g \cos \Theta_0 \phi$$
$$\dot{p} = \frac{I_{xz}}{I_{xx}} \dot{r} + L_v v + L_{\dot{v}} \dot{v} + L_r r + L_{\dot{r}} \dot{r} + L_p p + L_{\dot{p}} \dot{p} + L_{\delta_A} \delta_A + L_{\delta_R} \delta_R \tag{2.75}$$
$$\dot{r} = \frac{I_{xz}}{I_{zz}} \dot{p} + N_v v + N_{\dot{v}} \dot{v} + N_r r + N_{\dot{r}} \dot{r} + N_p p + N_{\dot{p}} \dot{p} + N_{\delta_A} \delta_A + N_{\delta_R} \delta_R$$

For conventional aircraft, it can usually be assumed that the following stability derivatives are insignificant:

$$Y_{\dot{v}},\ Y_p,\ Y_{\dot{p}},\ Y_r,\ Y_{\dot{r}},\ Y_{\delta_A},\ L_{\dot{v}},\ L_{\dot{r}},\ N_{\dot{v}},\ N_{\dot{r}}\ .$$

Note, however, that Y_r may be significant if U_0 is small. When this assumption is made the equations governing perturbed lateral/directional motion of the aircraft are given by:

$$\dot{v} = Y_v v - U_0 r + W_0 p + g \cos \Theta_0 \phi + Y_{\delta_R} \delta_R$$
$$\dot{p} = \frac{I_{xz}}{I_{xx}} \dot{r} + L_v v + L_p p + L_r r + L_{\delta_A} \delta_A + L_{\delta_R} \delta_R$$
$$\dot{r} = \frac{I_{xz}}{I_{zz}} \dot{p} + N_v v + N_p p + N_r r + N_{\delta_A} \delta_A + N_{\delta_R} \delta_R \tag{2.76}$$

$p = \dot{\phi} - \dot{\Psi} \sin \Theta_0$

$r = \dot{\Psi} \cos \Theta_0$

2.5 EQUATIONS OF MOTION IN STABILITY AXIS SYSTEM

The aerodynamic forces which contribute to the x, y and z terms in eq. (2.65) are the components of lift and drag resolved into the body-fixed axes. The angles which orient the forces of lift and drag relative to the body-fixed axes are: the angle of attack, α, and the angle of sideslip, β. The angles are defined in Figure 2.5 where the subscript 'α' has been used to indicate that the velocity and its components are relative in the sense of airframe to air mass. If the velocity of the air mass is constant relative to inertial space, then the subscript 'α' can be dropped. The velocity components along the body axes are:

$$\begin{aligned} U_\alpha &= V_{T_\alpha} \cos \beta \cos \alpha \\ V_\alpha &= V_{T_\alpha} \sin \beta \\ W_\alpha &= V_{T_\alpha} \cos \beta \sin \alpha \end{aligned} \tag{2.77}$$

Earlier it was shown that if symmetric flight was assumed, V_0 would be zero. Therefore, if the axis system is oriented such that W_0 is zero, then both α_0 and β_0 are zero. This orientation results in the X_B axis, in the steady state, pointing into the relative wind and the X_B axis and the velocity vector being aligned such that:

$$U_0 = V_T \tag{2.78}$$

Such an orientation results in a stability axis system which, initially, is inclined to the horizon at some flight path angle, γ_0, since:

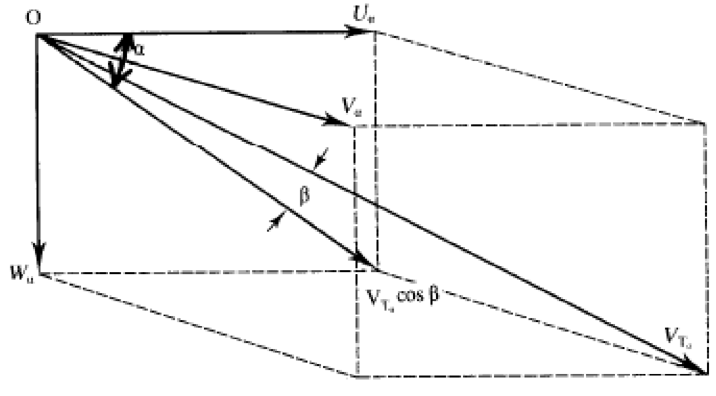


Figure 2.5 Orientation of relative wind with body axis system.

$$\Theta_0 \triangleq \gamma_0 + \alpha_0 \tag{2.79}$$

and α_0 is zero.

This initial alignment does not affect the body-fixed character of the axis system: all the motion due to perturbations is still measured in a body-fixed frame of reference. However, the alignment of the stability axis system with respect to the body axis system changes as a function of the trim conditions. When an aircraft is disturbed from its trim condition, the stability axes rotate with the airframe and, consequently, the perturbed X_s axis may or may not be parallel to the relative wind while the aircraft motion is being disturbed. The situation is illustrated in Figure 2.6.

Using the stability axis system, in which $W_0 = 0$ and $\Theta_0 = \gamma_0$, eq. (2.71) may be expressed as:

$$\begin{aligned}
\dot{u} &= X_u u + X_w w - g\cos\gamma_0\theta \\
\dot{w} &= Z_u u + Z_w w + U_0 q - g\sin\gamma_0\theta + Z_{\delta_E}\delta_E \\
\dot{q} &= M_u u + M_w w + M_{\dot{w}}\dot{w} + M_q q + M_{\delta_E}\delta_E \\
\dot{\theta} &= q
\end{aligned} \tag{2.80}$$

whereas eq. (2.76) may now be written as:

$$\dot{v} = Y_v v - U_0 r + g\cos\gamma_0\phi + Y_{\delta_R}\delta_R$$

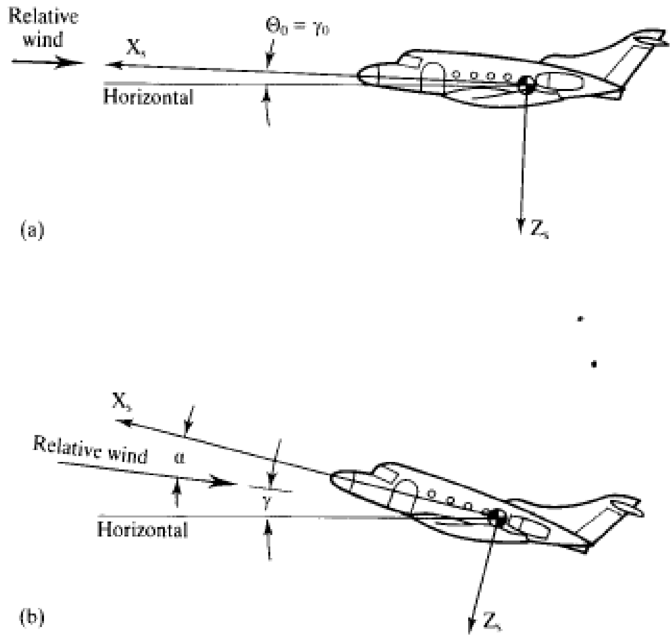


Figure 2.6 Direction of stability axes with respect to the relative wind. (a) Steady flight. (b) Perturbed flight.

$$\dot{p} = \frac{I_{xz}}{I_{xx}}\dot{r} + L_v v + L_p p + L_r r + L_{\delta_A}\delta_A + L_{\delta_R}\delta_R$$

$$\dot{r} = \frac{I_{xz}}{I_{zz}}\dot{p} + N_v v + N_p p + N_r r + N_{\delta_A}\delta_A + N_{\delta_R}\delta_R \quad (2.81)$$

$$\dot{\phi} = p + r\tan\gamma_0$$

$$\dot{\psi} = r/\cos\gamma_0$$

The cross-product inertia terms which appear in eq. (2.81) can be eliminated by a simple mathematical procedure: the use of primed stability derivatives. By ignoring second order effects, the cross-product of inertia terms are taken into account in the following primed stability derivatives:

$$L'_\beta = L_\beta + I_B N_\beta \qquad N'_\beta = N_\beta + I_A L_\beta$$

$$L'_p = L_p + I_B N_p \qquad N'_p = N_p + I_A L_p$$

$$L'_r = L_r + I_B N_r \qquad N'_r = N_r + I_A L_r \quad (2.82)$$

$$L'_{\delta_A} = L_{\delta_A} + I_B N_{\delta_A} \qquad N_{\delta_A} = N_{\delta_A} + I_A L_{\delta_A}$$

$$L'_{\delta_R} = L_{\delta_R} + I_B N_{\delta_R} \qquad N'_{\delta_R} = N_{\delta_R} + I_A L_{\delta_R}$$

in which

$$I_B \triangleq I_{xz}/I_{xx} \quad (2.83)$$

$$I_A \triangleq I_{xz}/I_{zz} \quad (2.84)$$

Then eq. (2.18) becomes

$$\dot{v} = Y_v v - U_0 r + g\cos\gamma_0\phi + Y_{\delta_R}\delta_R$$

$$\dot{p} = L'_v v + L'_p p + L'_r r + L'_{\delta_A}\delta_A + L'_{\delta_R}\delta_R$$

$$\dot{r} = N'_v v + N'_p p + N'_r r + N'_{\delta_A}\delta_A + N'_{\delta_R}\delta_R \quad (2.85)$$

$$\dot{\phi} = p + r\tan\gamma_0$$

$$\dot{\psi} = r\sec\gamma_0 = r/\cos\gamma_0$$

2.6 EQUATIONS OF MOTION FOR STEADY MANOEUVRING FLIGHT CONDITIONS

Steady flight conditions provide the reference values for many studies of aircraft motion. Once the relationships for steady flight are known, they are used subsequently to eliminate initial forces and moments from the equations of motion. How these steady relationships are determined is covered in the next sections.

2.6.1 Steady, Straight Flight

This is the simplest case of steady flight. All time derivatives are zero and there is no angular velocity about the centre of gravity. Therefore, setting to zero all time derivatives, the angular velocities P, Q, R, and the time derivatives of angular position (attitude) reduces eq. (2.56) to:

$$\begin{aligned} X_0 &= mg \sin\Theta \\ Y_0 &= -\,mg \cos\Theta \sin\Phi \\ Z_0 &= -\,mg \cos\Theta \cos\Phi \\ L_0 &= M_0 = N_0 = 0 \end{aligned} \tag{2.86}$$

These equations can be applied to a steady sideslip manoeuvre, for the velocity components V, W, and the bank angle, Φ, are not necessarily zero. However, if the motion is restricted to symmetric flight, the bank angle is zero. For this case, the equations become:

$$\begin{aligned} X_0 &= mg \sin\Theta \\ Y_0 &= 0 \\ Z_0 &= -\,mg \cos\Theta \end{aligned} \tag{2.87}$$

Again, all the moments are zero.

2.6.2 Steady Turns

In this case, the time derivatives are all zero again and the rates of change of the Euler angles, Φ and Θ, are also zero; the rate of turn, $\dot{\Psi}$, is constant. Generally, such steady, turning manoeuvres are carried out for very small pitching angles, or for shallow climbing or diving turns. Hence, for small θ, the following relationships hold (see eq. (2.46)):

$$\begin{aligned} P &= -\,\dot{\Psi}\, sin\,\Theta \simeq -\,\dot{\Psi}\Theta \\ Q &= \dot{\Psi} \cos\Theta \sin\Phi \simeq \dot{\Psi} \sin\Phi \\ R &= \dot{\Psi} \cos\Theta \cos\Phi \simeq \dot{\Psi} \cos\Phi \end{aligned} \tag{2.88}$$

For most manoeuvres of this type, $\dot{\Psi}$, although constant, is small so that the products of P, Q and R may be neglected. Furthermore, for co-ordinated shallow turns, the side force Y is zero (by definition) and the velocity components V and W are small. Therefore, for a steady, co-ordinated, shallow turn, the equations become:

$$\begin{aligned} X &= mg\Theta \\ Y &= +\,m(\dot{\Psi} U \cos\Phi - g \sin\Phi) \\ Z &= -\,mg\cos\phi \end{aligned} \tag{2.89}$$

$$\dot{\Psi} = \frac{g}{U_0} \tan \Phi$$

Again, all the moments are zero.

2.6.3 Steady Pitching Flight

Symmetric flight of an aircraft along a curved flight path, with constant pitching velocity Q, results in a quasi-steady flight condition. In this case, U and W do vary with time but V, P, R, Φ and Ψ are all zero. Therefore, the equations of motion for a rigid body aircraft reduce to:

$$\begin{aligned} X &= m(\dot{U} + QW) + mg \sin \Theta \\ Z &= m(\dot{W} - QU) - mg \cos \Theta \\ L &= M = N = Y = 0 \end{aligned} \tag{2.90}$$

Equation (2.90) can be used to evaluate the initial conditions which are used in the small perturbation analysis. For reasonable values of pitch rate, the linear accelerations $\dot{u}$ and $\dot{w}$ are negligibly small; consequently, eq. (2.90) becomes the initial conditions:

$$\begin{aligned} X_0 &= m(Q_0 W_0 + g \sin \Theta_0) \\ Z_0 &= -m(Q_0 U_0 + g \cos \Theta_0) \end{aligned} \tag{2.91}$$

If the second equation is solved, a relationship is obtained between the initial pitch rate Q_0 and the initial load factor n_{z_0}, along the Z_B axis:

$$\begin{aligned} Q_0 &= \frac{g}{U_0}\left(-\frac{Z_0}{mg} - \cos \Theta_0\right) \\ &= \frac{g}{U_0}(n_{z_0} - \cos \Theta_0) \end{aligned} \tag{2.92}$$

where

$$n_{z_0} = -Z_0/mg \tag{2.93}$$

2.6.4 Steady Rolling (Spinning) Flight

The equations of motion for steady rolling (spinning) flight cannot be simplified without improperly describing the physical situation so that the results obtained are unrepresentative of the actual motion. Special methods of treatment are required and, consequently, no such simplified equations are developed here. See, for example, Thelander (1965) for such methods.

2.7 ADDITIONAL MOTION VARIABLES

Even for the straightforward case of straight, steady, wings level, symmetric flight, the designer of AFCSs may be interested in motion variables other than the primary ones of change in forward speed u, in vertical velocity w, in pitch rate q, in pitch attitude θ, in sideslip velocity v, in roll rate p, in yaw rate r, in bank angle ϕ, and in yaw angle ψ. Other commonly used motion variables are treated here, with particular regard to the development of their relationship to the primary motion variables. Such additional motion variables are usually those which can be measured by the sensors commonly available on aircraft.

2.7.1 Longitudinal Motion

Normal acceleration, for perturbed motion, and measured at the c.g. of the aircraft, is defined as:

$$a_{z_{cg}} = (\dot{w} - U_0 q) \tag{2.94}$$

For small angles of attack, α,

$$w \simeq U_0 \alpha$$

$$\therefore \quad a_{z_{cg}} = U_0(\dot{\alpha} - q) \tag{2.95}$$

In aircraft applications, acceleration is often measured in units of g, in which case

$$n_{z_{cg}} = \frac{a_{z_{cg}}}{g} \tag{2.96}$$

When an aircraft changes its attitude, the steady, normal acceleration due to gravity, g, also changes. In that case:

$$a_{z_{cg}} = \dot{w} - U_0 q - g \tag{2.97}$$

If it is required to know the acceleration at some point, x distant from the c.g. by l_x, but still on the fuselage centre line, that acceleration is given by:

$$a_{z_x} = \dot{w} - U_0 q - l_x \dot{q} \tag{2.98}$$

The distance l_x from the c.g. is measured positive forwards. By definition:

$$\ddot{h}_{cg} = -a_{z_{cg}} \tag{2.99}$$

where h is the height of the aircraft's c.g. above the ground. Consequently:

$$\dot{h}_{cg} = -w + U_0\theta \tag{2.100}$$

$$h_{cg} = U_0 \int \theta dt - \int w dt$$
$$= U_0 \int \gamma dt \tag{2.101}$$

$$\therefore \quad n_{z_{cg}} = -U_0\dot{\gamma}/g \tag{2.102}$$

The variation of load factor with the angle of attack of an aircraft, n_{z_α}, is an important aircraft parameter known as the acceleration sensitivity. It will be shown in Chapter 3 how n_{z_α} can be determined from the stability derivatives and the equations of motion; the result obtained there is quoted here for convenience:

$$n_{z_\alpha} = \frac{U_0}{g} \frac{(Z_{\delta_E}M_w - M_{\delta_E}Z_w)}{\left(M_{\delta_E} - Z_{\delta_E}\frac{M_q}{U_0}\right)} \simeq \frac{U_0}{gM_{\delta_E}}(Z_{\delta_E}M_w - M_{\delta_E}Z_w) \tag{2.103}$$

Usually, for conventional aircraft, $M_{\delta_E}Z_w \gg Z_{\delta_E}M_w$; consequently:

$$n_{z_\alpha} = -Z_wU_0/g \tag{2.104}$$

For straight and level flight, at $n_z = 1$

$$n_{z_\alpha} = -Z_wU_0/g = C_{L_\alpha}/C_L \tag{2.105}$$

where C_{L_α} is the lift curve slope and C_L is the coefficient of lift.

2.7.2 Lateral Motion

In lateral motion, the perturbed acceleration at the c.g. of the aircraft is defined by:

$$a_{y_{cg}} \triangleq \dot{v} - g\phi + U_0r \tag{2.106}$$

If it is required to know the lateral acceleration at some point, x_{lat}, on the OX axis, distant from the c.g. by $l_{x_{lat}}$, and displaced a distance, l_z, on the OZ axis, the appropriate equation is:

$$a_{y_{x_{lat}}} = a_{y_{cg}} + l_{x_{lat}}\dot{r} - l_z\dot{p} \tag{2.107}$$

$l_{x_{lat}}$ is measured positive forwards of the c.g. and l_z is measured positive downwards. Heading angle, λ, is defined as the sum of sideslip, β, and yaw angle, Ψ.

$$\lambda = \Psi + \beta \tag{2.107a}$$

2.8 THE STATE AND OUTPUT EQUATIONS

2.8.1 The State Equation

A state equation is a first order, vector differential equation. It is a natural form in which to represent the equation of motion of an aircraft. Its most general expression is:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{2.108}$$ [4]

where $\mathbf{x} \in R^n$ is the state vector, $\mathbf{u} \in R^m$ is the control vector.

The elements of the vector $\mathbf{x}$ are termed the state variables and the elements of the vector $\mathbf{u}$ the control input variables. A is the state coefficient matrix and B the driving matrix; they are of order $(n \times n)$ and $(n \times m)$, respectively.

From an inspection of eq. (2.108) it should be observed that the l.h.s. terms involve only first derivatives of the state variables with respect to time; the r.h.s. depends solely upon the state vector $\mathbf{x}$ and the control vector $\mathbf{u}$. Thus, the state equation is an attractive mathematical form for aircraft control and stability studies since its solution for known inputs can easily be obtained by means of integration. Furthermore, this same form of equation lends itself to simulation. In Chapter 1 it was stated that the flight of an aircraft can be affected as much by disturbances such as atmospheric turbulence as by deliberate control inputs, $\mathbf{u}$. Such disturbances can be taken into account by adding a term to the r.h.s. of eq. (2.108), i.e.:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + E\mathbf{d} \tag{2.109}$$

where $\mathbf{d}$ is a vector of dimension l which represents the l sources of disturbance. The associated matrix, E, is of order $(n \times l)$. If the disturbances are random, special methods are used to introduce the disturbances into the aircraft's state equation which is generally considered to be deterministic. These methods are dealt with separately in Chapter 5, and, consequently, for the remainder of this chapter $\mathbf{d}$ will be regarded as a null vector.

Any set of first order, linear, constant coefficient, ordinary differential equations can be combined into the form of eq. (2.108).

2.8.2 The Output Equation

If the concern is with motion variables other than those chosen as state variables, then an output equation is wanted. The output equation is merely an algebraic equation which depends solely upon the state vector, and, occasionally, upon the control vector also. Its customary form of expression is:

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u} \tag{2.110}$$ [5]

The output vector is $\mathbf{y} \in R^p$ and its elements are referred to as the output variables. The matrices C and D, the output and direct matrix respectively, are generally rectangular and are of order $(p \times n)$ and $(p \times m)$, respectively.

For AFCS work the sensors used to measure motion variables, for use as feedback signals, are often subject to measurement noise. To incorporate these noise effects into an output equation requires the addition of another term to eq. (2.110):

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u} + F\boldsymbol{\xi} \tag{2.111}$$ [6]

The characterization of sensor noise and how it is modelled dynamically are dealt with in Chapter 5. For the rest of this present chapter $\boldsymbol{\xi}$ is assumed to be null.

2.8.3 Aircraft Equations of Longitudinal Motion

If the state vector is defined as, say:

$$\mathbf{x} = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} \tag{2.112}$$

and if an aircraft is being controlled only by means of elevator deflection, δ_E, such that its control vector is defined as:

$$\mathbf{u} \triangleq \delta_E \tag{2.113}$$

then, from eq. (2.80):

$$A \triangleq \begin{bmatrix} X_u & X_w & 0 & -g\cos\gamma_0 \\ Z_u & Z_w & U_0 & -g\sin\gamma_0 \\ \tilde{M}_u & \tilde{M}_w & \tilde{M}_q & \tilde{M}_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.114}$$

$$B \triangleq \begin{bmatrix} X_{\delta_E} \\ Z_{\delta_E} \\ \tilde{M}_{\delta_E} \\ 0 \end{bmatrix} \tag{2.115}$$

The significance of the tilde in row 3 of eq. (2.114) is easily explained. In eq. (2.80) the equation for $\dot{q}$ was written as:

$$\dot{q} = M_u u + M_w w + M_{\dot{w}}\dot{w} + M_q q + M_{\delta_E}\delta_E \tag{2.116}$$

It is obvious that a term in $\dot{w}$ exists on the r.h.s. of the equation. The state equation, though, does not admit on its r.h.s. terms involving the first (or even higher) derivatives of any of the state or control variables. Fortunately, $\dot{w}$, itself, depends only upon $\mathbf{x}$ and $\mathbf{u}$ and, therefore, an easy substitution is possible. In eq. (2.80) the equation for $\dot{w}$ is given as: Z_{δ_E}

$$\dot{w} = Z_u u + Z_w w + U_0 q - g\sin\gamma_0\theta + M_{\delta_E}\delta_E \tag{2.117}$$

Substituting for $\dot{w}$ in the equation for $\dot{q}$ yields:

$$\dot{q} = (M_u + M_{\dot{w}}Z_u)u + (M_w + M_{\dot{w}}Z_w)w$$

$$+ (M_q + M_{\dot{w}}U_0)q - gM_{\dot{w}} \sin \gamma_0 \theta + (M_{\delta_E} + M_{\dot{w}} Z_{\delta_E})\delta_E$$

i.e.

$$\dot{q} = \bar{M}_u u + \bar{M}_w w + \bar{M}_q q + \bar{M}_\theta \theta + \bar{M}_{\delta_E} \delta_E \qquad (2.118)$$

where

$$\begin{aligned} \bar{M}_u &= (M_u + M_{\dot{w}} Z_u) \\ \bar{M}_w &= (M_w + M_{\dot{w}} Z_w) \\ \bar{M}_q &= (M_q + U_0 M_{\dot{w}}) \\ \bar{M}_\theta &= (-\, gM_{\dot{w}} \sin \gamma_0) \\ \bar{M}_{\delta_E} &= (M_{\delta_E} + M_{\dot{w}} Z_{\delta_E}) \end{aligned} \qquad (2.119)$$

If there were some other control inputs on the aircraft being considered, say, for example, a change of thrust, δ_{th}, and a deflection of symmetrical spoilers, δ_{sp}, then the order of the driving matrix, B, becomes (4×3) and the elements of the matrix become:

$$B = \begin{bmatrix} X_{\delta_E} & X_{\delta_{th}} & X_{\delta_{sp}} \\ Z_{\delta_E} & Z_{\delta_{th}} & Z_{\delta_{sp}} \\ \bar{M}_{\delta_E} & \bar{M}_{\delta_{th}} & \bar{M}_{\delta_{sp}} \\ 0 & 0 & 0 \end{bmatrix} \qquad (2.120)$$

It must be understood that the state equation is not an unique description of the aircraft dynamics. For example, if the state vector had been chosen to be

$$\mathbf{x} \triangleq \begin{bmatrix} \theta \\ q \\ u \\ w \end{bmatrix} \qquad (2.121)$$

rather than the choice of eq. (2.112), A and B must be changed to:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \bar{M}_u & \bar{M}_w & \bar{M}_q & \bar{M}_\theta \\ X_u & X_w & 0 & -g \cos \gamma_0 \\ Z_u & Z_w & U_0 & -g \sin \gamma_0 \end{bmatrix} \qquad (2.122)$$

$$B = \begin{bmatrix} 0 \\ \bar{M}_{\delta_E} \\ X_{\delta_E} \\ Z_{\delta_E} \end{bmatrix} \qquad (2.123)$$

When the state equation is solved, with either set of A and B, the responses obtained for the same control input, δ_E, will be identical.

In American work it is common to use as a primary motion variable the angle of attack, α, rather than the heave velocity, w. Since, for small angles:

$$\alpha = w/U_0 \tag{2.124}$$

then:

$$\dot{\alpha} = \dot{w}/U_0 \tag{2.125}$$

$$\mathrm{d}\alpha U_0 = \mathrm{d}w \tag{2.126}$$

$$\therefore \quad \dot{\alpha} = \frac{Z_u}{U_0} u + Z_w \frac{w}{U_0} + q + \frac{Z_{\delta_E}}{U_0} \delta_E \tag{2.127}$$

$$= Z_u^* u + Z_w \alpha + q + Z_{\delta_E}^* \delta_E \tag{2.127}$$

where

$$Z_u^* = Z_u/U_0 \text{ and } Z_{\delta_E}^* = Z_{\delta_E}/U_0 .$$

Frequently, again in American papers, a stability derivative Z_α is quoted, and eq. (2.127) is written as:

$$\dot{\alpha} = Z_u^* u + Z_\alpha \alpha + q + Z_{\delta_E}^* \delta_E \tag{2.128}$$

The reader is warned, however, that confusion can occur with this form. In eq. (2.128) Z_α is identical to Z_w in eq. (2.127), but, for consistency of notation, Z_α ought to be defined as:

$$Z_\alpha \triangleq \partial Z/\partial \alpha = Z_w U_0 \tag{2.129}$$

Z_α is sometimes quoted as a value which turns out to be identical to Z_w, and sometimes as equal to $Z_w U_0$. The student is advised *always* to use the form of equation given in (2.117) and from the state equation obtain the heave velocity w. If the angle of attack is required, then determine α from eq. (2.124). In this way, ambiguity and confusion can be avoided.

If the output variable of interest was, say, a_{z_x}, then eq. (2.98) can easily be shown (by substitution for w and q) to be given by:

$$a_{z_x} = (Z_u - l_x\tilde{M}_u)u + (Z_w - l_x\tilde{M}_w)w - l_x\tilde{M}_q q + (Z_{\delta_E} - l_x\tilde{M}_{\delta_E})\delta_E \tag{2.130}$$

Hence:

$$y \triangleq a_{z_x} = [(Z_u - l_x\tilde{M}_u)(Z_w - l_x\tilde{M}_w) - l_x\tilde{M}_q \;\; 0]\mathbf{x} + [(Z_{\delta_E} - l_x\tilde{M}_{\delta_E})]\mathbf{u} \tag{2.131}$$

which is the same form as eq. (2.110), where

$$C = [(Z_u - l_x\tilde{M}_u)(Z_w - l_x\tilde{M}_w) - l_x\tilde{M}_q \quad 0] \tag{2.132}$$

$$D = (Z_{\delta_E} - l_x\tilde{M}_{\delta_E}) \tag{2.133}$$

If the concern is with the height of an aircraft at its c.g., then:

$$\ddot{h}_{cg} = -a_{z_{cg}} \tag{2.99}$$

$$a_{z_{cg}} = Z_u u + Z_w w + Z_{\delta_E}\delta_E \tag{2.134}$$

i.e.

$$\ddot{h} = -Z_u u - Z_w w - Z_{\delta_E}\delta_E$$

To express this in terms of state variables let:

$$x_6 = h \tag{2.135}$$

and let:

$$x_5 = \dot{x}_6 = \dot{h} \tag{2.136}$$

$$\therefore \quad \dot{x}_5 = -Z_u u - Z_w w - Z_{\delta_E}\delta_E \tag{2.137}$$

Hence:

$$\mathbf{x} \triangleq \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \\ \dot{h} \end{bmatrix} \text{ and } \mathbf{u} = [\delta_E] \tag{2.138}$$

Then the state equation (2.108) is obtained once more, i.e.:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{2.108}$$

but now:

$$A = \begin{bmatrix} X_u & X_w & 0 & -g\cos\gamma_0 & 0 & 0 \\ Z_u & Z_w & U_0 & -g\sin\gamma_0 & 0 & 0 \\ \tilde{M}_u & \tilde{M}_w & \tilde{M}_q & \tilde{M}_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -Z_u & -Z_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.139}$$

$$B = \begin{bmatrix} X_{\delta_E} \\ Z_{\delta_E} \\ \tilde{M}_{\delta_E} \\ 0 \\ -Z_{\delta_E} \\ 0 \end{bmatrix} \tag{2.140}$$

If the motion variable being considered is the flight path angle γ then it can be inferred from eq. (2.79) that:

$$\gamma = \theta - \alpha = \theta - (w/U_0) \tag{2.141}$$

Consequently, if $\mathbf{y} \triangleq \gamma$, then

$$\mathbf{y} = \begin{bmatrix} 0 & -\frac{1}{U_0} & 0 & 1 \end{bmatrix} \mathbf{x} = C\mathbf{x} \tag{2.142}$$

where $\mathbf{x}$ is defined as in eq. (2.112).

2.8.4 Aircraft Equations of Lateral Motion

For lateral motion, the control vector may be defined as:

$$\mathbf{u} \triangleq \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix} \tag{2.143}$$

If the state vector, $\mathbf{x}$, is defined as:

$$\mathbf{x} \triangleq \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} \tag{2.144}$$

then the state equation is given by:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{2.108}$$

where:

$$A = \begin{bmatrix} Y_v & 0 & -U_0 & +g\cos\gamma_0 & 0 \\ L'_v & L'_p & L'_r & 0 & 0 \\ N'_v & N'_p & N'_r & 0 & 0 \\ 0 & 1 & \tan\gamma_0 & 0 & 0 \\ 0 & 0 & \sec\gamma_0 & 0 & 0 \end{bmatrix} \tag{2.145}$$

$$B = \begin{bmatrix} 0 & Y_{\delta_R} \\ L'_{\delta_A} & L'_{\delta_R} \\ N'_{\delta_A} & N'_{\delta_R} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.146}$$

The sideslip angle, β, is often used as a state variable, rather than the sideslip velocity, v. From eq. (2.77), for small angles:

$$v = U_0\beta \tag{2.147}$$

and consequently:

$$\dot{\beta} = Y_v\beta - r + \frac{g}{U_0}\cos\gamma_0\phi + \frac{Y_{\delta_R}}{U_0}\delta_R \tag{2.148}$$

which may be written as:

$$\dot{\beta} = Y_v\beta - r + \frac{g}{U_0}\cos\gamma_0\phi + Y^*_{\delta_R}\delta_R \tag{2.149}$$

where:

$$Y^*_{\delta_R} = Y_{\delta_R}/U_0 \tag{2.150}$$

If, now, the state vector is defined as:

$$\mathbf{x} = \begin{bmatrix} \beta \\ p \\ r \\ \phi \\ \psi \end{bmatrix} \tag{2.151}$$

then eq. (2.108) obtains, but the coefficient matrix has become:

$$A = \begin{bmatrix} Y_v & 0 & -1 & \frac{g}{U_0}\cos\gamma_0 & 0 \\ L'_\beta & L'_p & L'_r & 0 & 0 \\ N'_\beta & N'_p & N'_r & 0 & 0 \\ 0 & 1 & \tan\gamma_0 & 0 & 0 \\ 0 & 0 & \sec\gamma_0 & 0 & 0 \end{bmatrix} \tag{2.152}$$

The driving matrix has become:

$$B = \begin{bmatrix} 0 & Y^*_{\delta_R} \\ L'_{\delta_A} & L'_{\delta_R} \\ N'_{\delta_A} & N'_{\delta_R} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.153}$$

The fifth column of A in both eqs (2.145) and (2.152) is composed entirely of zeros. The physical significance of this is explained in Chapter 3, but the presence of such a column of zeros can often be avoided by redefining the state vector, as in eq. (2.154) which has now dimension 4; i.e. let:

$$\mathbf{x} = \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} \tag{2.154}$$

then A becomes:

$$A = \begin{bmatrix} Y_v & 0 & -1 & g/U_0 \\ L'_\beta & L'_p & L'_r & 0 \\ N'_\beta & N'_p & N'_r & 0 \\ 0 & 1 & \tan\gamma_0 & 0 \end{bmatrix} \tag{2.155}$$

and B becomes:

$$B = \begin{bmatrix} 0 & Y^*_{\delta_R} \\ L'_{\delta_A} & L'_{\delta_R} \\ N'_{\delta_A} & N'_{\delta_R} \\ 0 & 0 \end{bmatrix} \tag{2.156}$$

It must be emphasized that in straight and level flight (i.e. non-climbing or diving) γ_0 is zero. Consequently, for this flight condition, those elements which appear in the various forms of A, and which depend upon γ_0, will take a value of zero if the element has the form $\sin\gamma_0$ or $\tan\gamma_0$, or will take the value unity if the element involves $\cos\gamma_0$ or $\sec\gamma_0$. Sometimes there is interest in the lateral acceleration of an aircraft at some point x, which is a distance l_x from the c.g. (l_x is positive forwards) and a distance l_z off the axis OX (l_z is positive when down from the c.g.). Hence:

$$a_{y_x} = a_{y_{cg}} + l_x\dot{r} - l_z\dot{p} \tag{2.107}$$

which can easily be shown to be:

$$\begin{aligned} a_{y_x} = (Y_v + l_xN'_v - l_zL'_v)v + (l_xN'_p - l'_zL'_p)p \\ + (l_xN'_r - l_zL'_r)r + (l_xN'_{\delta_A} - l_zL'_{\delta_A})\delta_A \\ + (Y^*_{\delta_R} + l_xN'_{\delta_R} - l_zL'_{\delta_R})\delta_R \end{aligned} \tag{2.157}$$

If the output variable $\mathbf{y}$ is taken as the lateral acceleration, then eq. (2.157) can be expressed as:

$$\mathbf{y} = [(Y_v + l_xN'_v - l_zL'_v)(l_xN'_p - l_zL'_p)(l_xN'_r - l_zL'_r) \quad 0]\mathbf{x}$$

$$+ [(l_x N'_{\delta_A} - l_z L'_{\delta_A})(Y^*_{\delta_R} + l_x N'_{\delta_R} - l_z L'_{\delta_R})]\mathbf{u}$$
$$= C\mathbf{x} + D\mathbf{u} \tag{2.158}$$

2.9 OBTAINING A TRANSFER FUNCTION FROM STATE AND OUTPUT EQUATIONS

Whenever the variables of a linear system are expressed in the complex frequency domain, i.e. as functions of the Laplace variable s, then, whenever the initial conditions can be assumed to be zero, the ratio of the output variable to some particular input variable (all other input variables being considered identically zero) is the transfer function of the system.

Given that the small perturbation dynamics of an aircraft can be represented by a state equation of the form of eq. (2.108) and an output equation of the form of eq. (2.110), namely $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ and $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$ respectively, then, provided that $\mathbf{y}$ is scalar and that only those columns of matrices B and D are used which correspond to the particular control input u_j being considered, then a transfer function relating $\mathbf{y}$ and u_j can be found. If $\mathbf{y}$ is a vector and it is required to find the transfer function corresponding to some particular element, y, as a result of some control input, u_j, the rows of the matrices C and D which correspond to y_i are used in the calculation. To illustrate the procedure consider that $\mathbf{y}$ and $\mathbf{u}$ are scalars. Taking Laplace transforms, and assuming initial conditions are zero, results in eqs (2.108) and (2.110) being expressed as:

$$s\mathbf{X}(s) - A\mathbf{X}(s) = BU(s) \tag{2.159}$$

$$\mathrm{y}(s) = C\mathbf{X}(s) + DU(s) \tag{2.160}$$

$$\therefore \quad \mathbf{X}(s) = (sI - A)^{-1}\, BU(s)$$

$$\therefore \quad y(s) = [C(sI - A)^{-1}\, B + D]U(s) \tag{2.161^7}$$

$$\therefore \quad y(s)/U(s) \triangleq G(s) = C[sI - A]^{-1}\, B + D \tag{2.162}$$

In general, if:

$$G(s) = y_i(s)/u_j(s) \tag{2.163}$$

then:

$$G(s) = C_i[sI - A]^{-1}\, B_j + D_{ij} \tag{2.164}$$

where B_j represents the column of matrix B which corresponds to u_j, and D_{ij} is the ith row of the matrix D corresponding to y_i and the jth column corresponding to u_j. C_i is the ith row of matrix C corresponding to y_i.

It is evident that transfer function relationships can be found for output motion caused by sensor noise or by atmospheric disturbances rather than manoeuvre commands acting through the control inputs, but these are not treated until Chapter 5.

2.10 IMPORTANT STABILITY DERIVATIVES

All stability derivatives are important but some are more important for flight control than others. This section treats only the latter type.

A number of parameters appear frequently in the equations defining stability derivatives. They are listed here for convenience (note that all the stability derivatives presented are dimensional): S is the surface area of the wing, $\bar{c}$ is the mean aerodynamic chord, ρ is the density, and b is the wing span.

2.10.1 Longitudinal Motion

Motion-related

$$M_u \triangleq \frac{\rho S U_0 \bar{c}}{I_{yy}} (C_{m_u} + C_m) \qquad (2.165)[8]$$

The non-dimensional pitching moment coefficient C_m is usually zero in trimmed flight, except in cases of thrust asymmetry. M_u represents the change in pitching moment caused by a change in forward speed. Its magnitude can vary considerably and its sign can change with changes in Mach number and in dynamic pressure and also as a result of aeroelastic effects. In modern aircraft, the Mach number effects and the effects of aeroelasticity have become increasingly important. ∎

$$Z_w = -\frac{\rho S U_0}{2m} (C_{L_\alpha} + C_D) \qquad (2.166)$$

The change in lift coefficient with a change in angle of attack, C_{L_α}, is often referred to as the lift curve slope. It is always positive for values of angle of attack below the stall value. The lift curve slope for the total airframe comprises components due to the wing, the fuselage and the tail. For most conventional aircraft it has been found to be generally true that the wing contributes 85–90 per cent to the value of C_{L_α}. Consequently, any aeroelastic distortion of the wing can appreciably alter C_{L_α} and, hence, Z_w. ∎

$$M_w = \frac{\rho S U_0 \bar{c}}{2 I_{yy}} C_{m_\alpha} \qquad (2.167)$$

The non-dimensional stability derivative, C_{m_α}, is the change in the pitching moment coefficient with angle of attack. It is referred to as the 'longitudinal static stability derivative'. C_{m_α} is very much affected by any aeroelastic distortions of the wing, the tail and the fuselage. However, both sign and magnitude of C_{m_α} are principally affected by the location of the c.g. of the aircraft. C_{m_α} is proportional to the distance, x_{AC}, between the c.g. and the aerodynamic centre (a.c.) of the whole aircraft. x_{AC} is measured positive forwards. If x_{AC} is zero, C_{m_α} is zero. If $x_{AC} < 0$, C_{m_α} is negative and the aircraft is statically stable. If the c.g. is aft of the a.c., $x_{AC} < 0$ and C_{m_α} is positive, with the consequence that the aircraft is statically unstable. In going from subsonic to supersonic flight the a.c. generally moves aft, and, therefore, if the c.g. remains fixed, C_{m_α} will tend to increase for a statically stable aircraft. $M_w(M_\alpha)$ is closely related to the aircraft's static margin. The significance of stability, static margin and M_w, is discussed in section 3.3 of Chapter 3, but it can be stated simply here that M_w (or M_α) is the most important longitudinal derivative. ■

$$M_{\dot{w}} = \frac{\rho S \bar{c}^2}{4 I_{yy}} C_{m_{\dot{\alpha}}} \tag{2.168}$$

Although $C_{m_{\dot{\alpha}}}$ does not have a powerful effect upon an aircraft's motion, particularly the short period motion, it does have a significant effect. Usually $M_{\dot{w}} < 0$; it increases the damping of the short period motion. ■

$$M_q = \frac{\rho S U_0 \bar{c}^2}{4 I_{yy}} C_{m_q} \tag{2.169}$$

For conventional aircraft, M_q contributes a substantial part of the damping of the short period motion. This damping comes mostly from changes in the angle of attack of the tail and it is also proportional to the tail length, l_{T}. But l_{T} is the lever arm through which the lift force on the horizontal tail is converted into a moment, i.e.: ← this "alpha" means "proportional"

$$M_q \alpha l_{\mathrm{T}}^2 \tag{2.170}$$

M_q is a very significant stability derivative which has a primary effect on the handling qualities of the aircraft (see Chapter 6). ■

Control-related

$$Z_{\delta_{\mathrm{E}}} = \frac{-\rho U_0^2 S}{2m} C_{L_{\delta_{\mathrm{E}}}} \tag{2.171}$$

Since $C_{L_{\delta_{\mathrm{E}}}}$ is usually very small, $Z_{\delta_{\mathrm{E}}}$ is normally unimportant except when an AFCS involving feedback of normal acceleration is used. Also, if a tailless aircraft is being considered, the effective lever arm for the elevator (or ailerons) is small, hence $C_{L_{\delta_{\mathrm{E}}}}$ may be relatively large compared to $C_{m_{\delta_{\mathrm{E}}}}$. In these cases, $Z_{\delta_{\mathrm{E}}}$ cannot safely be neglected in any analysis. ■

$$M_{\delta_{\mathrm{E}}} = \frac{\rho U_0^2 S \bar{c}}{2 I_{yy}} C_{m_{\delta_{\mathrm{E}}}} \tag{2.172}$$

$C_{m_{\delta_{\mathrm{E}}}}$ is termed the 'elevator control effectiveness'; it is very important in aircraft design and for AFCS work. When the elevator is located aft of the c.g.,[9] the normal location, $C_{m_{\delta_{\mathrm{E}}}}$ is negative. Its value is determined chiefly by the maximum lift of the wing and also the range of c.g. travel which can occur during a flight. ■

2.10.2 Lateral Motion

Motion-related

$$Y_v = \frac{\rho U_0 S}{2m} C_{y_\beta} \tag{2.173}$$

The sideforce which results from any sideslip motion is usually obtained from the fin of the aircraft, and usually opposes the sideslip motion, i.e. $C_{y_\beta} < 0$. But for aircraft with a slender fuselage, at large values of the angles of attack the forces can be in an aiding direction. For certain (rare) configurations having a wing of low aspect ratio but required to operate at a large value of angle of attack, this force on the fuselage can counter the resisting force of the fin which results in the stability derivative C_{y_β} being positive. Such positive values, even if very small, are undesirable because the reversed (or small) side force makes it difficult for a pilot to detect sideslip motion and consequently makes a co-ordinated turn difficult to achieve. Such values of C_{y_β} also reduce the damping ratio of the dutch roll mode, whereas C_{y_β} normally makes a large contribution to this damping. In the normal case C_{y_β} is not a derivative which causes great difficulty to AFCS designers. ■

$$L_\beta = U_0 L_v = \frac{\rho U_0^2 S b}{2 I_{xx}} C_{l_\beta} \tag{2.174}$$

Note that:

$$L'_\beta = \frac{L_\beta + (I_{xz}/I_{xx}) N_\beta}{1 - (I_{xz}^2/I_{xx} I_{zz})} \tag{2.175}$$

The change in the value of the rolling moment coefficient with sideslip angle C_{l_β} is called the 'effective dihedral'. This derivative is very important in studies concerned with lateral stability and control. It features in the damping of both the dutch roll and the spiral modes. It also affects the manoeuvring capability of an aircraft, particularly when lateral control is being exercised near stall by rudder action only. Usually small negative values of C_{l_β} are wanted, as such values improve the damping of both the dutch roll and the spiral modes, but such values are rarely obtained without considerable aerodynamic difficulty. ■

$$N_\beta = \frac{\rho U_0^2}{2} \frac{Sb}{I_{xx}} C_{n_\beta} \tag{2.176}$$

The change in the yawing moment coefficient with change in sideslip angle C_{n_β} is referred to as the 'static directional' or 'weathercock' stability coefficient. It depends upon the area of the fin and the lever arm. The aerodynamic contribution to C_{n_β} from the fin is positive, but the contribution from the aircraft body is negative. A positive value of C_{n_β} is regarded as static directional stability; a negative value signifies static directional instability (see Chapter 3). C_{n_β} primarily establishes the natural frequency of the dutch roll mode and is an important factor in establishing the characteristics of the spiral mode stability. For good handling qualities C_{n_β} should be large, although such values magnify the disturbance effects from side gusts. At supersonic speeds C_{n_β} is adversely affected because the lift curve slope of the fin decreases. ■

$$L_p = \frac{\rho U_0 S b^2}{4I_{xx}} C_{l_p} \tag{2.177}$$

The change in rolling moment coefficient with change in rolling velocity, C_{l_p} is referred to as the roll damping derivative. Its value is determined almost entirely by the geometry of the wing. In conjunction with $C_{l_{\delta_A}}$ (q.v.), C_{l_p} establishes the maximum rolling velocity which can be obtained from the aircraft: an important flying quality. C_{l_p} is always negative, although it may become positive when the wing (or parts of it) are stalled. ■

$$N_p = \frac{\rho U_0 S b^2}{4I_{xx}} C_{n_p} \tag{2.178}$$

The change in ~~rolling~~ yawing moment coefficient with a change in rolling velocity, C_{n_p}, is usually negative, although a positive value is desirable. The more negative is C_{n_p} the smaller is the damping ratio of the dutch roll mode and the greater is the sideslip motion which accompanies entry to, or exit from, a turn. ■

$$L_r = \frac{\rho U_0 S b^2}{4I_{xx}} C_{l_r} \tag{2.179}$$

The change in rolling moment coefficient with a change in yawing velocity, C_{l_r}, has a considerable effect on the spiral mode, but does not much affect the dutch roll mode. For good spiral stability, C_{l_r} should be positive but as small as possible. A major contributing factor to C_{l_r} is the lift force from the wing, but if the fin is located either above or below the axis OX it also makes a substantial contribution to C_{l_r}, being positive or negative dependent upon the fin's geometry. ■

$$N_r = \frac{\rho U_0 S b^2}{4I_{zz}} C_{n_r} \tag{2.180}$$

The change in yawing moment coefficient with a change in yawing velocity, C_{n_r}, is referred to as the 'yaw damping derivative'. It is proportional to l_T^2. Usually C_{n_r} is negative and is the main contributor to the damping of the dutch roll mode. It also contributes to the stability of the spiral mode. ■

Control-related

$$Y_\delta = \frac{\rho U_0^2 S}{2m} C_{y_\delta} \tag{2.181}$$

The change in side force coefficient with rudder deflection, $C_{y_{\delta_R}}$, is unimportant *except* when considering an AFCS using lateral acceleration as feedback. $C_{y_{\delta_A}}$ is nearly always negligible. Because positive rudder deflection produces a positive side force, $C_{y_{\delta_R}} < 0$. ■

$$L_\delta = \frac{\rho U_0^2 Sb}{2I_{xx}} C_{l_\delta} \tag{2.182}$$

$C_{l_{\delta_R}}$ is the change in rolling moment coefficient which results from rudder deflection. It is usually negligible. Because the rudder is usually located above the axis OX, positive rudder deflection produces positive rolling motion, i.e. $C_{l_{\delta_R}} > 0$.

The change in rolling moment coefficient with a deflection of the ailerons, $C_{l_{\delta_A}}$, is referred to as the *aileron effectiveness*. In lateral dynamics it is the most important control-related stability derivative. It is particularly important for low speed flight where adequate lateral control is needed to counter asymmetric gusts which tend to roll the aircraft. ■

$$N_\delta = \frac{\rho U_0^2 Sb}{2I_{zz}} C_{n_\delta} \tag{2.183}$$

The change in yawing moment coefficient which results from a rudder deflection, $C_{n_{\delta_R}}$, is referred to as the *rudder effectiveness*. When the rudder is deflected to the left (i.e. $\delta_R > 0$) a negative yawing moment is created on the aircraft, i.e. $C_{n_{\delta_R}} < 0$.

The change in yawing moment coefficient which results from an aileron deflection, $C_{n_{\delta_A}}$, results in *adverse yaw* if $C_{n_{\delta_A}} < 0$, for when a pilot deflects the ailerons to produce a turn, the aircraft will yaw initially in a direction opposite to that expected. When $C_{n_{\delta_A}} > 0$ the yaw which results is favourable to that turning manoeuvre, and this is referred to as *proverse yaw*. Whatever sign $C_{n_{\delta_A}}$ takes, its value ought to be small for good lateral control. ■

2.11 THE INCLUSION OF THE EQUATIONS OF MOTION OF THRUST EFFECTS

1. Many of the stability derivatives which are used in the equations of motion are the result not only of aerodynamic forces but of forces arising from flows induced by the propulsion system. Such flows profoundly modify the derivatives but the effects are usually difficult to predict,

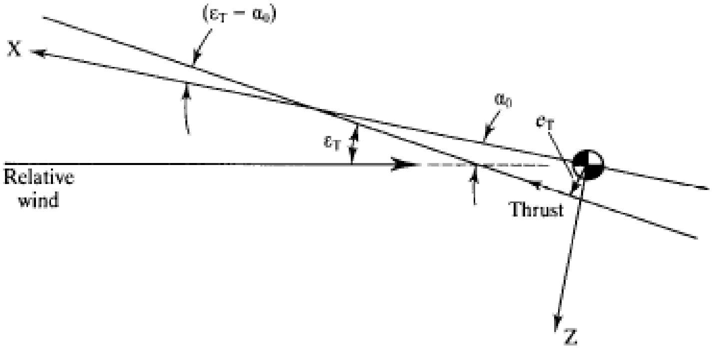


Figure 2.7 Thrust alignment geometry.

requiring special wind tunnel tests for their resolution. But where slipstream interference is minimal, such being the case when a subsonic jet has a central exhaust aft of the tail, the forces and moments associated with direct thrust make considerable contributions to various derivatives. The number of forces associated with the propulsion system include:

(a) The forces acting on the inlet which result when the air mass entering the engine changes direction.
(b) The moments caused by the angular velocity of a tube containing a mass of moving air.
(c) The forces and moments resulting from the thrust itself.

2. The angle which the thrust line makes with the relative wind is ε_T (see Figure 2.7) and is fixed by both the geometry of the aircraft and its trim condition. The angle of the thrust line with respect to the X-axis is fixed at $(\varepsilon_T - \alpha_0)$. Hence:

$$X_T = T\cos(\varepsilon_T - \alpha_0) \tag{2.184}$$

$$Z_T = -T\sin(\varepsilon_T - \alpha_o) \tag{2.185}$$

$$M_T = e_T T \tag{2.186}$$

where the thrust offset e_T is positive downwards.

3. Of course, thrust is a function of density, throttle setting, and the relative speed of the aircraft (on rare occasions it is a function of α_0). Hence:

$$dX_T = \cos(\varepsilon_T - \alpha_0)\left\{\frac{\partial T}{\partial V}\left(\frac{\partial V}{\partial U}u + \frac{\partial V}{\partial W}w\right) + \frac{\partial T}{\partial \delta_{th}}\delta_{th}\right\} \tag{2.187}$$

$$dZ_T = \sin(\varepsilon_T - \alpha_0)\left\{\frac{\partial T}{\partial V}\left(\frac{\partial V}{\partial U}u + \frac{\partial V}{\partial W}w\right) + \frac{\partial T}{\partial \delta_{th}}\delta_{th}\right\} \tag{2.188}$$

However:

$$\frac{\partial X_T}{\partial U} = \frac{\partial T}{\partial V}(\cos\varepsilon_T\cos^2\alpha_0 + \sin\varepsilon_T\sin\alpha_0\cos\alpha_0) \tag{2.189}$$

$$\frac{\partial X_T}{\partial W} = \frac{\partial T}{\partial V}(\cos \varepsilon_T \sin \alpha_0 \cos \alpha_0 + \sin \varepsilon_T \sin^2 \alpha_0) \qquad (2.190)$$

$$\frac{\partial X_T}{\partial \delta_{th}} = \frac{\partial T}{\partial \delta_{th}}(\cos \varepsilon_T \cos \alpha_0 + \sin \varepsilon_T \sin \alpha_0) \qquad (2.191)$$

$$\frac{\partial Z}{\partial U} = \frac{-\partial T}{\partial V}(\sin \varepsilon_T \cos^2 \alpha_0 - \cos \varepsilon_T \sin \alpha_0 \cos \alpha_0) \qquad (2.192)$$

$$\frac{\partial Z_T}{\partial W} = \frac{-\partial T}{\partial V}(\sin \varepsilon_T \sin \alpha_0 \cos \alpha_0 - \cos \varepsilon_T \sin^2 \alpha_0) \qquad (2.193)$$

$$\frac{\partial Z_T}{\partial \delta_{th}} = \frac{-\partial T}{\partial \delta_{th}}(\sin \varepsilon_T \cos \alpha_0 - \cos \varepsilon_T \sin \alpha_0) \qquad (2.194)$$

At the trim condition, however, the total moment must be zero, i.e. the thrust moment must be balanced by an equal and opposite aerodynamic moment. Thus:

$$M_0 = T_0 e_T + \frac{\rho U_0^2}{2} S\bar{c}C_m = 0 \qquad (2.195)$$

$$\begin{aligned} dM = e_T \left\{ \frac{\partial T}{\partial V}\left(\frac{\partial V}{\partial U}u + \frac{\partial V}{\partial W}w\right) + \frac{\partial T}{\partial \delta_{th}}\delta_{th} \right\} \\ + \rho U_0 S\bar{c}C_m\left(\frac{\partial V}{\partial U}u + \frac{\partial V}{\partial W}w\right) \end{aligned} \qquad (2.196)$$

From eq. (2.195), however:

$$T_0 e_T = \frac{-\rho U_0^2 S\bar{c}}{2} C_m \qquad (2.197)$$

i.e.

$$\rho U_0 S\bar{c}C_m = \frac{-2T_0 e_T}{U_0} \qquad (2.198)$$

$$\therefore \quad dM = e_T \left\{ \left(\frac{\partial T}{\partial V} - \frac{2T_0}{U_0}\right)(u \cos \alpha_0 + w \sin \alpha_0) + \frac{\partial T}{\partial \delta_{th}}\delta_{th} \right\} \qquad (2.199)$$

It is evident that the perturbations in moment due to thrust are influenced by the trim condition term, T_0/U_0.

4. Thrust can be written as:

$$T = \frac{\rho U_0^2}{2} SC_{th} \qquad (2.200)$$

However, C_{th} is *not* an aerodynamic coefficient so that eq. (2.200) is misleading. The thrust contribution manifests itself chiefly in X_u and is expressed in the form:

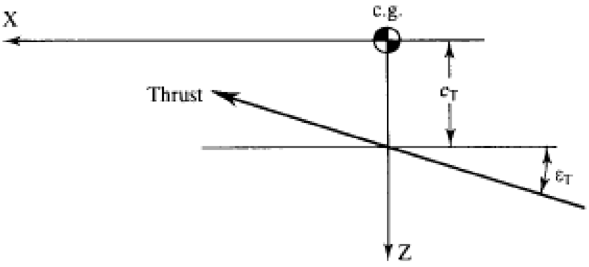


Figure 2.8 Resolution of thrust into forces and moments.

$$X_u = \frac{-\rho S U_0}{m}\left(\frac{U_0}{2}\frac{\partial C_D}{\partial U} + C_D\right) + \frac{1}{m}\frac{\partial T_x}{\partial U} \tag{2.201}$$

where T_x is the component of thrust along the axis OX. The partial derivative $\partial T_x/\partial U$ is found from data on the power plant. The direct contribution of thrust to other stability derivatives is usually negligible.

5. When the throttle setting, δ_{th}, is increased there is a corresponding increase in thrust. Figure 2.8 shows how thrust is resolved into forces and moments.

From Figure 2.8:

$$X_{\delta_{th}} = \frac{1}{m}\left(\frac{\partial T}{\partial \delta_{th}}\right)\cos \varepsilon_T \tag{2.202}$$

$$Z_{\delta_{th}} = -\frac{1}{m}\left(\frac{\partial T}{\partial \delta_{th}}\right)\sin \varepsilon_T \tag{2.203}$$

$$M_{\delta_{th}} = -\left(\frac{e_T}{I_{yy}}\right)\left(\frac{\partial T}{\partial \delta_{th}}\right)\cos \varepsilon_T \tag{2.204}$$

2.12 CONCLUSIONS

The form of the equations of motion of an aircraft depends upon the axis system which has been chosen. Once a particular axis system is adopted, it is helpful to expand the aerodynamic force and moment terms, and to linearize the inertial and gravitational terms so that when small perturbations are considered the resulting equations will be linear and can be separated into longitudinal and lateral motion. Using the stability axis system is the most convenient for AFCS work. Sometimes, small motion is not of concern, however, and it is essential instead to consider steady manoeuvring flight such as pitching or turning. Not every motion variable of interest appears in the resulting equations of motion; such important variables as flight path angle, height, heading, and normal and lateral accelerations, are related, however, to these equations and this chapter shows how these variables can be obtained from a knowledge of the equations of motion. The form of the equation lends itself to representing the longitudinal and lateral dynamics of the aircraft directly as state equations, with the other variables being obtained from associated output equations. Once the state and output equations are known it is possible to determine any transfer function relating a particular output variable to a particular control input.

Not every stability derivative is significant in terms of its influence on the dynamics of the aircraft and only the most important need to be studied for their likely effects on the subsequent performance of an AFCS. Thrust changes do affect the motion of an aircraft, of course, but the thrust line does not always act through the c.g. of the aircraft, the origin of the stability axis system upon which the equations of motion are based. Consequently, special techniques are needed to introduced these thrust effects into the equations of motion.

2.14 NOTES

1. For example, see chapter 4 of McRuer *et al.* (1973).
2. This depends upon the assumption of constant aircraft mass.

3. m_1 has been used to denote the perturbation in the pitching moment, M, to avoid confusion with the aircraft's mass, m.
4. This form applies to linear, time-invariant systems only; when the system is non-linear, the appropriate form is $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$.
5. For linear, time invariant systems only; when the output relationship is non-linear the appropriate form is $\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$.
6. If the output equation is non-linear, the presence of measurement noise modifies $\mathbf{y}$ to become: $\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \boldsymbol{\xi}, t)$.
7. This assumes that the matrix $(sI - A)$ is non-singular, which can be proved by recalling that $\mathcal{L}^{-1}\{[sI - A]^{-1}\} = e^{At}$.
8. Although U_0 is used in these equations, the correct value to be used is the true airspeed. For small perturbations, the errors are insignificant if U_0 is used instead of V_T.
9. If the elevator is located forward of the c.g. it is renamed *canard*. This description is increasingly common, although canard referred originally to an aircraft configuration which flew 'tail first', the forward tail surface being called a foreplane. It is this foreplane which is now considered to be a canard.

2.15 REFERENCES

BABISTER, A.W. 1961. *Aircraft Stability and Control.* Oxford: Pergamon Press.

FOGARTY, L.E. and R.M. HOWE. 1969. Computer mechanization of six-degree-of-freedom flight equations. NASA CR-1344, May.

GAINES, T.G. and S. HOFFMAN. 1972. Summary of transformation equations and equations of motion. NASA Sp-3070.

McRUER, D.T., I.L. ASHKENAS and D.C. GRAHAM. 1973. *Aircraft Dynamics and Automatic Control.* Princeton University Press.

McRUER, D.T., C.L. BATES and I.L. ASHKENAS. 1953. Dynamics of the airframe. Bur. Aero. Rpt. AE-61-4 (Vol. II) USA.

THELANDER, J.A. 1965. Aircraft motion analysis. FDL-TDR-67-70, WPAFB, Ohio, USA. March.

3

Aircraft Stability and Dynamics

3.1 INTRODUCTION

The equations of motion have been derived in some detail in Chapter 2. Only under a large number of assumptions about how an aircraft is being flown is it possible to arrive at a set of linear differential equations which can adequately represent the motion that results from the deflection of a control surface or from the aircraft's encountering atmospheric turbulence during its flight. This resulting motion is composed of small perturbations about the equilibrium (trim) values. To achieve such equilibrium values requires the use of certain steady deflections of the appropriate control surfaces. Consequently, the entire range of the angle of deflection of any particular control surface will not necessarily be available for the purposes of automatic control, since much of that range is required to trim the aircraft. What is meant, then, by small perturbation is that any angle be sufficiently small to guarantee that the assumptions concerning any trigonometrical functions involved remain valid. For practical purposes, a change of angle of 15° or more should be regarded as large, and the designer should then consider the likely effects of continuing to use the small perturbation theory whenever such angular values can occur. Similarly, translational velocity should always be small in relation to the steady speeds; when the steady speed, such as V_0 or W_0, is zero then changes of velocity of $5\,\mathrm{m\,s^{-1}}$ should be regarded as being the limit of validity. However, it must be strongly emphasized that these are not firm rules but depend upon the type of aircraft being considered, its flight condition, and the manoeuvres in which it is involved.

For the remainder of this chapter it is considered that all the assumptions of Chapter 2 hold, that any aircraft being considered is fixed wing and flying straight and level in a trimmed condition, and that its motion is properly characterized by eqs (2.109) and (2.110). For example, for longitudinal motion, eq. (2.112) is taken as the definition of the state vector **x**, i.e.:

$$\mathbf{x} \triangleq \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} \tag{2.112}$$

and the control vector **u** is defined as:

$$\mathbf{u} \triangleq [\delta_E] \tag{2.113}$$

The state coefficient matrix A is then given by:

$$A = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & U_0 & 0 \\ \tilde{M}_u & \tilde{M}_w & \tilde{M}_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.1}$$

and the driving matrix B by:

$$B = \begin{bmatrix} X_{\delta_E} \\ Z_{\delta_E} \\ \tilde{M}_{\delta_E} \\ 0 \end{bmatrix} \tag{3.2}$$

For lateral motion, the appropriate equations are (2.143) and (2.154), respectively where the coefficient matrix is:

$$A = \begin{bmatrix} Y_v & 0 & -1 & g/U_0 \\ L'_\beta & L'_p & L'_r & 0 \\ N'_\beta & N'_p & N'_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{3.3}$$

and the driving matrix is:

$$B = \begin{bmatrix} Y^*_{\delta_A} & Y^*_{\delta_R} \\ L'_{\delta_A} & L'_{\delta_R} \\ N'_{\delta_A} & N'_{\delta_R} \\ 0 & 0 \end{bmatrix} \tag{3.4}$$

3.2 LONGITUDINAL STABILITY

3.2.1 Short Period and Phugoid Modes

The dynamic stability of perturbed longitudinal motion is most effectively established from a knowledge of the eigenvalues of the coefficient matrix A. They can be found by solving the linear equation:

$$|\lambda I - A| = 0 \tag{3.5}$$

I is a 4×4 identity matrix. By expanding the determinant, the longitudinal stability quartic, a fourth degree polynomial in λ, can be expressed as:

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{3.6}$$

An aircraft may be said to be dynamically stable if all its eigenvalues, λ_i, being real, have negative values, or, if they be complex, have negative real parts. Zero, or positive, values of the real part of any complex eigenvalue means that the aircraft will be dynamically unstable.[1] Rather than solving the polynomial by numerical methods it is more effective to use a numerical routine to compute the four eigenvalues of A.

It has been observed that for the majority of aircraft types, the quartic of eq. (3.6) invariably factorizes into two quadratic factors in the following manner:

$$(\lambda^2 + 2\zeta_{ph}\omega_{ph}\lambda + \omega_{ph}^2)(\lambda^2 + 2\zeta_{sp}\omega_{sp}\lambda + \omega_{sp}^2) \tag{3.7}$$

The first factor corresponds to a mode of motion which is characterized by an oscillation of long period. The damping of this mode is usually very low, and is sometimes negative, so that the mode is unstable and the oscillation grows with time. The low frequency associated with the long period motion is defined as the natural frequency, ω_{ph}; the damping ratio has been denoted as ζ_{ph}. The mode is referred to as the *phugoid* mode, a name improperly given to it by the English aerodynamicist, Lanchester, who coined it from the Greek word which he believed meant 'flight-like'. Unfortunately, φυγη implies flight as demonstrated by a fugitive, not a bird (Sutton, 1949). The second factor corresponds to a rapid, relatively well-damped motion associated with the short period mode whose frequency is ω_{sp} and damping ration is ζ_{sp}.

As an example, consider the passenger transport aircraft, referred to as aircraft DELTA in Appendix B. If flight condition 4 is considered, the aircraft is flying straight and level in its cruise phase, at Mach 0.8 and at a height of 13 000 m. From the values of the stability derivatives quoted in the appendix, A is found to be:

$$A = \begin{bmatrix} -0.033 & 0.0001 & 0 & -9.81 \\ 0.168 & -0.387 & 260.0 & 0 \\ 55 \times 10^{-4} & -0.0064 & -0.551 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.8}$$

The eigenvalues corresponding to this matrix are found to be:

$$\lambda_1, \lambda_2 = +0.0033 \pm j0.0672 \qquad (3.9)^2$$

$$\lambda_3, \lambda_4 = -0.373 \pm j0.889 \qquad (3.10)^2$$

The eigenvalues of eq. (3.9) are seen to be those associated with the phugoid mode since the damping ratio, although positive, is very small (0.0489) and the frequency is very low (0.067 rad s^{-1}), hence the period is long. Such an inference can be drawn because the solution of any quadratic equation of the form:

$$x^2 + 2\zeta\omega x + \omega^2 = 0 \tag{3.11}$$

is given by:

$$\begin{aligned} x_1 &= -\zeta\omega + j\omega\sqrt{(1-\zeta^2)} \\ x_2 &= -\zeta\omega - j\omega\sqrt{(1-\zeta^2)} \end{aligned} \tag{3.12}$$

whenever $\zeta < 1.0$. Complex roots occur only when the damping ratio has a positive value less than unity.

From eq. (3.10) the eigenvalues can be deduced to be those associated with the short period mode, for which the frequency is 0.964 rad s^{-1} and the damping ratio is 0.387.

3.2.2 Tuck Mode

Supersonic aircraft, or aircraft which fly at speeds close to Mach 1.0, occasionally have a value of the stability derivative, M_u, such that M_u takes a large value which is sufficiently negative to result in the term ω_{ph}^2 in the phugoid quadratic becoming negative too (see Section 3.6). When this happens, the roots of the quadratic equation are both real, with one being negative and the other positive. Hence the phugoid mode is no longer oscillatory but has become composed of two real modes; one being convergent, which corresponds to the negative real root, and the other being divergent, which corresponds to the positive real root. The unstable mode is referred to as the 'tuck mode' because the corresponding motion results in the nose of the aircraft dropping (tucking under) as airspeed increases. Aircraft DELTA in Appendix B will exhibit a divergent tuck mode in flight condition 3.

3.2.3 A Third Oscillatory Mode

The c.g. of a modern combat aircraft is often designed to lie aft of the neutral point (n.p.) (see Section 3.3). When this is the case the stability derivative, M_w, can take a value which will result in every root of the longitudinal stability quartic being real. As the c.g. is then moved further aft of the n.p., the value of M_w changes so that one of the real roots of the short period mode, and one of the real roots of the phugoid mode, migrate in the complex plane to a point where they form a new complex pair, corresponding to the third oscillatory mode. When this has occurred, that mode is the main influence upon the dynamic response of any AFCS which is used. The phugoid mode has now become a very slow aperiodic mode, and there also exists another extremely rapid real mode. Too positive a value of M_w can result in dynamic instability, for one of these real eigenvalues can become positive (see Section 3.5.2).

3.2.4 s-plane Diagram

The location of eigenvalues in the complex frequency domain is often represented by means of an s-plane diagram (which is simply a special Argand diagram). In Figure 3.1 are shown the locations (denoted by ×) of eigenvalues for a typical conventional aircraft. For an aircraft which exhibits a tuck mode the locations are denoted by ○ and for an aircraft with a third oscillatory mode they are denoted by △.

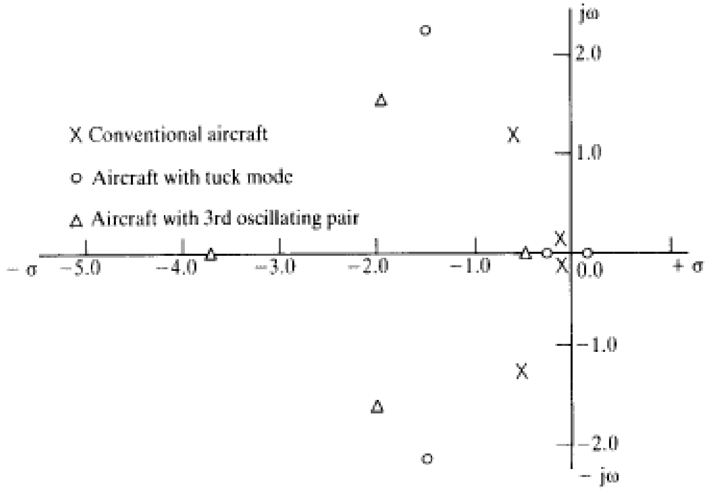


Figure 3.1 s-plane diagram.

A popular method of investigating how sensitive is an aircraft's stability to values of some particular stability derivative (and, consequently, some aerodynamic, inertial, or geometric parameter) is to illustrate how the eigenvalues travel around the s-plane as the values of the stability derivative are changed. This is a form of root locus diagram. Another effective way of determining to which stability derivative the aircraft's dynamic response is most sensitive is to carry out a sensitivity analysis on coefficient matrix, A (Barnett and Storey, 1966). It is important to remember that when the aircraft dynamics can be assumed to be linear those stability derivatives associated with the control surfaces play no part in governing the stability properties of the aircraft. Their importance for achieving effective automatic flight control, including stability augmentation, is paramount nevertheless.

3.4 TRANSFER FUNCTIONS RELATED TO LONGITUDINAL MOTION

3.4.1 Relationship Between Transfer Function and State Equation

The theory relating to deriving transfer functions from the linearized equations of motion is given in Section 2.9 of Chapter 2. In this present section, some of the more commonly used transfer functions for longitudinal motion will be derived, but the reader should be aware that a number of computer programs are available (see for example, Systems Control Technology, Inc., 1986; Larimer, 1978) for the automatic determination of appropriate transfer functions from a knowledge of the stability derivatives. These programs are usually based on the Leverrier algorithm (Faddeeva, 1959).

The purpose of deriving analytically a number of transfer functions in this present section is to arrive at their final forms, to see which parameters and terms are significant, and to note possible simplifications which can lead to useful approximations.

It has been shown in Chapter 2 that if only a single control, δ_E, is considered, the linearized, small perturbation equations of longitudinal motion are given by:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{3.23}$$

where:

$$\mathbf{x} \triangleq \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} \tag{3.24}$$

$$\mathbf{u} \triangleq [\delta_E] \tag{3.25}$$

The coefficient matrix, A, and the driving matrix, B, are given by:

$$A = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & U_0 & 0 \\ \tilde{M}_u & \tilde{M}_w & \tilde{M}_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.26}$$

$$B = \begin{bmatrix} X_{\delta_E} \\ Z_{\delta_E} \\ \tilde{M}_{\delta_E} \\ 0 \end{bmatrix} \tag{3.27}$$

From eq. (2.164), the transfer function relating output variable, y_i, to control input, u_i, is given by:

$$G(s) = C_i[sI - A]^{-1} B_j + D_{ij} \qquad (2.164)$$

Thus, every transfer function depends upon the variable chosen as the output and the control surface deflection used to change the motion variable. But it must always be remembered that when the control deflection is used to change some particular motion variable that same control deflection changes other motion variables simultaneously. It is this simple fact which sometimes causes great difficulty for the designers of AFCSs, and it is this fact which results in so many systems, designed by means of the conventional theory of control for single input, single output, linear systems, producing aircraft performance which is unacceptable to pilots. Although transfer functions are useful, their use is limited, particularly for AFCS design for modern aircraft where many control surfaces are employed simultaneously. However, from eq. (2.164) it is evident that every transfer function relating to the motion of the aircraft must depend on the inherent characteristics of the aircraft through the resolvent matrix, $[sI - A]^{-1}$.

3.4.2 Use of Output Matrix, C, to Select a Particular Motion Variable

For the present, normal acceleration, and those motion variables such as h which are directly related to it, are not being considered. Thus:

$$\mathbf{y} = C\mathbf{x} \qquad (3.28)$$

and, for further simplicity, since transfer functions are being considered, only a single output variable will be dealt with at a time. Consequently, eq. (3.28) now becomes:

$$y = C\mathbf{x} \qquad (3.29)$$

where C is a 1×4 rectangular matrix. C contains only one non-zero element and that element has the value unity. The column in which this value is to be found depends upon which state variable is being taken as the output variable of concern. For example, if the output variable is chosen to be u, then:

$$y \triangleq [1 \;\; 0 \;\; 0 \;\; 0]\mathbf{x} \qquad (3.30)$$

The other three relationships are:

$$y \triangleq w = [0 \;\; 1 \;\; 0 \;\; 0]\mathbf{x} \qquad (3.31)$$

$$y \triangleq q = [0 \;\; 0 \;\; 1 \;\; 0]\mathbf{x} \qquad (3.32)$$

$$y \triangleq \theta = [0 \;\; 0 \;\; 0 \;\; 1]\mathbf{x} \qquad (3.33)$$

Thus, the unit element can be looked upon as a kind of pointer indicating which state variable has been chosen as the output variable.

Quite often, the output matrix C is used to achieve conversion of physical units. For example, if the state variable q is defined in rad s^{-1} but is required to work with pitch rate in degree s^{-1}, then defining q in degree s^{-1} as an output variable results in $y = [0 \;\; 0 \;\; 57.3 \;\; 0]\mathbf{x}$.

3.4.3 Transfer Function Notation

It will be plain to the reader now that four transfer functions can be determined, namely:

$$u(s)/\delta_E(s),\ w(s)/\delta_E(s),\ q(s)/\delta_E(s) \text{ and } \theta(s)/\delta_E(s)$$

The form of these transfer functions is identical:

$$G(s) = N(s)/D(s) \tag{3.34}$$

The denominator polynomial is the characteristic polynomial of the aircraft, namely $\det[\lambda I - A]$ which was dealt with in Section 3.2. When the roots of the polynomial are known, i.e. those values of s are known which result in:

$$\Delta_{\text{long}}(s) = \det[sI - A] = 0 \tag{3.35}$$

it will be seen that they are identical to the eigenvalues of A. The polynomial $\det[sI - A]$ is often called the stability quartic. Every transfer function for longitudinal motion has the same denominator, because every transfer function must represent the characteristic motion of the same aircraft. Therefore, the only way in which the transfer functions can differ for a particular motion, longitudinal or lateral, of an aircraft, is in their numerator polynomials. These numerator polynomials are direct functions of the output variable and the control input, and to emphasize this fact, they are often denoted, in American reports especially, as $N_{uj}^{yi}(s)$. The superscript yi denotes the particular output variable, and uj denotes the control input. Thus, for the four transfer functions considered up to this point, the corresponding denotations would be: $N_{\delta_E}^{u}(s)$, $N_{\delta_E}^{w}(s)$, $N_{\delta_E}^{q}(s)$, and $N_{\delta_E}^{\theta}(s)$.

For longitudinal motion the matrix $[sI - A]^{-1}$ can be shown to be:

$$[sI - A]^{-1} \triangleq \frac{\operatorname{adj}[sI - A]}{\det[sI - A]} = \frac{\begin{bmatrix} n_{11}(s) & n_{12}(s) & n_{13}(s) & n_{14}(s) \\ n_{21}(s) & n_{22}(s) & n_{23}(s) & n_{24}(s) \\ n_{31}(s) & n_{32}(s) & n_{33}(s) & n_{34}(s) \\ n_{41}(s) & n_{42}(s) & n_{43}(s) & n_{44}(s) \end{bmatrix}}{(s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4)} \tag{3.36}$$

The elements, n_{11}, of the numerator matrix are given as follows:

$$n_{11}(s) = s\{s^2 - [M_q + M_{\dot{w}}U_0 + Z_w]s + [Z_w M_q - M_w U_0]\} \tag{3.37}$$

$$n_{12}(s) = X_w s^2 - X_w[M_q + M_{\dot{w}}U_0]s - g[M_w + M_{\dot{w}}Z_w] \tag{3.38}$$

$$n_{13}(s) = s(U_0 X_w - g) + gZ_w \tag{3.39}$$

$$n_{21}(s) = Z_u s^2 - [Z_u M_q - M_u U_0]s = s[Z_u s - (Z_u M_q - M_u U_0)] \tag{3.40}$$

$$n_{22}(s) = s^3 - [X_u + M_q + M_{\dot{w}}U_0]s^2 + X_u[M_q + M_{\dot{w}}U_0]s + g[M_u + M_{\dot{w}}Z_u] \tag{3.41}$$

$$n_{23}(s) = U_0 s^2 - X_u U_0 s - gZ_u \tag{3.42}$$

$$n_{31}(s) = s\{s[M_u + M_{\dot{w}}Z_u] + [Z_uM_{\dot{w}} - Z_wM_u]\} \quad (3.43)$$

$$n_{32}(s) = s^2[M_w + M_{\dot{w}}Z_w] - s[X_uM_w - X_wM_u + M_{\dot{w}}[Z_wX_u - Z_uX_w)] \quad (3.44)$$

$$n_{33}(s) = s\{s^2 - [X_u + Z_w]s + [X_uZ_w - Z_uX_w]\} \quad (3.45)$$

$$n_{41}(s) = s[M_u + M_{\dot{w}}Z_u] + [Z_uM_w - M_uZ_w] \quad (3.46)$$

$$n_{42}(s) = s[M_w + M_{\dot{w}}Z_w] + Z_wM_u - X_uM_w + M_{\dot{w}}[Z_uX_w - X_uZ_w] \quad (3.47)$$

$$n_{43}(s) = s^2 - [X_u + Z_w]s + [X_uZ_w - Z_uX_w] \quad (3.48)$$

The elemental functions, $n_{14}(s)$, $n_{24}(s)$, $n_{34}(s)$ and $n_{44}(s)$ are all identically zero (because the fourth element in the driving matrix, B, is zero, i.e. $b_{41} = 0$). The coefficients, a_i, of the characteristic polynomial can be determined by evaluating $\det[sI - A]$. They are:

$$a_1 = -(X_u + M_q + Z_w + M_{\dot{w}}U_0) \quad (3.49)$$

$$a_2 = (M_qZ_w - M_wU_0 + X_uZ_w - Z_uX_w + X_uM_q + X_uU_0M_{\dot{w}}) \quad (3.50)$$

$$a_3 = -(X_uZ_wM_q + X_uM_wU_0 - M_qZ_uX_w + M_uX_wU_0 - gM_u - gM_{\dot{w}}Z_u) \quad (3.51)$$

$$a_4 = g(Z_uM_w - Z_wM_u) \quad (3.52)$$

Thus, firstly:

$$\frac{u(s)}{\delta_E(s)} = \frac{n_{11}(s)b_{11} + n_{12}(s)b_{21} + n_{13}(s)b_{31}}{\Delta_{long}(s)}$$

$$\therefore \quad \frac{u(s)}{\delta_E(s)} = \frac{N^u_{\delta_E}(s)}{\Delta_{long}(s)} = \frac{b_3s^3 + b_2s^2 + b_1s + b_0}{s^4 + a_1s^3 + a_2s^2 + a_3s + a_4} \quad (3.54)$$

where:

$$b_3 = X_{\delta_E} \quad (3.55)$$

$$b_2 = -X_{\delta_E}[Z_w + M_q + M_{\dot{w}}U_0] + Z_{\delta_E}X_w \quad (3.56)$$

$$b_1 = X_{\delta_E}[Z_wM_q - M_wU_0] - Z_{\delta_E}[X_wM_q + gM_{\dot{w}}] + M_{\delta_E}[X_wU_0 - g] \quad (3.57)$$

$$b_0 = g[M_{\delta_E}Z_w - Z_{\delta_E}M_w] \quad (3.58)$$

The a_i are defined in eqs (3.49)–(3.52).

Secondly:

$$\frac{w(s)}{\delta_E(s)} = \frac{N^w_{\delta_E}(s)}{\Delta_{long}(s)} = \frac{\hat{b}_3s^3 + \hat{b}_2s^2 + \hat{b}_1s + \hat{b}_0}{\Delta_{long}(s)} \quad (3.59)$$

where:

$$\hat{b}_3 = Z_{\delta_E} \quad (3.60)$$

$$\hat{b}_2 = X_{\delta_E}Z_u - Z_{\delta_E}[X_u + M_q] + M_{\delta_E}U_0 \quad (3.61)$$

$$\hat{b}_1 = X_{\delta_E}[U_0M_u - Z_uM_q] + X_u[Z_{\delta_E}M_q - U_0M_{\delta_E}] \quad (3.62)$$

$$\hat{b}_0 = g[Z_{\delta_E}M_u - M_{\delta_E}Z_u] \tag{3.63}$$

Thirdly:

$$\frac{q(s)}{\delta_E(s)} = \frac{N^q_{\delta_E}(s)}{\Delta_{long}(s)} = \frac{s\{b'_2s^2 + b'_1s + b'_0\}}{\Delta_{long}(s)} \tag{3.64}$$

where

$$b'_2 = [M_{\delta_E} + M_{\dot{w}}Z_{\delta_E}] \tag{3.65}$$

$$b'_1 = X_{\delta_E}[M_u + M_{\dot{w}}Z_u] + Z_{\delta_E}[M_w - M_{\dot{w}}X_u] - M_{\delta_E}[X_u + Z_w] \tag{3.66}$$

$$\begin{aligned} b'_0 = {} & X_{\delta_E}[Z_uM_w - Z_wM_u] + Z_{\delta_E}[X_wM_u - X_uM_w] \\ & + M_{\delta_E}[X_uZ_w - Z_uX_w] \end{aligned} \tag{3.67}$$

Note that knowing eq. (3.64) means that $\theta(s)/\delta_E(s)$ is known:

$$\theta(s)/\delta_E(s) = (b'_2s^2 + b'_1s + b'_0)/\Delta_{long}(s) \tag{3.68}$$

3.4.4 Transfer Functions Involving Motion Variables Other Than State Variables

It has been shown how the four primary transfer functions relating to longitudinal motion can be evaluated. Other longitudinal transfer functions can be as easily found. For example, since it is known that:

$$\alpha = w/U_0 \tag{3.69}$$

then:

$$\frac{\alpha(s)}{\delta_E(s)} = \frac{1}{U_0}\frac{w(s)}{\delta_E(s)} = \frac{\hat{b}_3s^3 + \hat{b}_2s^2 + \hat{b}_1s + \hat{b}_0}{U_0[\Delta_{long}(s)]}$$

$h(s)/\delta_E(s)$ can be evaluated by making use of eq. (2.94):

$$a_{z_{cg}} = \dot{w} - U_0q = -\ddot{h} \tag{2.94}$$

$$\therefore \quad -\frac{s^2h(s)}{\delta_E(s)} = \frac{sw(s)}{\delta_E(s)} - U_0\frac{q(s)}{\delta_E(s)} \tag{3.71}$$

$$\therefore \quad -\frac{sh(s)}{\delta_E(s)} = \frac{w(s)}{\delta_E(s)} - U_0\frac{\theta(s)}{\delta_E(s)} \tag{3.72}$$

$$\begin{aligned} \therefore \quad -\frac{sh(s)}{\delta_E(s)} &= \frac{\hat{b}_3s^3 + \hat{b}_2s^2 + \hat{b}_1s + \hat{b}_0}{\Delta_{long}(s)} - \frac{U_0(b'_1s^2 + b'_1s + b'_0)}{\Delta_{long}(s)} \\ &= \frac{\hat{b}_3s^3 + \hat{b}_2s^2 + \hat{b}_1s + \hat{b}_0}{\Delta_{long}(s)} \end{aligned} \tag{3.73}$$

where:

$$\hat{b}_3 = \not{b} = Z_{\delta_E} \tag{3.74}$$

$\hat{b}_3$

$$\hat{b}_2 = (\hat{b}_2 - U_0 b_2') = X_{\delta_E} Z_u - Z_{\delta_E}(X_u + M_q + M_{\dot{w}} U_0) \tag{3.75}$$

$$\begin{aligned}\hat{b}_1 = (\hat{b}_1 - U_0 b_1') = &- X_{\delta_E} Z_u [M_q + M_{\dot{w}} U_0] \\ &+ Z_{\delta_E}[X_u (M_q + U_0 M_{\dot{w}}) - U_0 M_w] + M_{\delta_E} U_0 Z_w\end{aligned} \tag{3.76}$$

$$\begin{aligned}\hat{b}_0 = (\hat{b}_0 - U_0 b_0') = &X_{\delta_E}(U_0 Z_w M_u - U_0 M_w Z_u] - Z_{\delta_E}[M_u (U_0 X_w - g) \\ &- M_w U_0 X_u] - M_{\delta_E}[U_0 Z_w X_u + Z_u (g - X_w U_0)]\end{aligned} \tag{3.77}$$

3.4.5 Numerical Example

Using the numerical data presented in Appendix B for aircraft BRAVO at flight condition 1, it is easy to determine that the characteristic equation is given by:

$$\Delta_{long}(s) = (s^4 + 2.92 s^3 + 2.178 s^2 + 0.015 s + 0.01) = 0 \tag{3.78}$$

which can be factorized as:

$$\Delta_{long}(s) = (s^2 + 0.00068 s + 0.0046)(s^2 + 2.9136 s + 2.17) = 0 \tag{3.79}$$

Then:

$$\frac{u(s)}{\delta_E(s)} = \frac{- 0.003 s^2 + 0.435 s + 0.48}{(s^4 + 2.92 s^3 + 2.178 s^2 + 0.015 s + 0.01)} \tag{3.80}$$

$$\frac{w(s)}{\delta_E(s)} = \frac{- 95.166(s^3 + 85.426 s^2 + 1.9717 s + 80.86)}{(s^4 + 2.92 s^3 + 2.178 s^2 + 0.015 s + 0.01)} \tag{3.81}$$

$$\frac{q(s)}{\delta_E(s)} = \frac{- 13.04(s^2 + 0.707 s + 0.01)}{(s^4 + 2.92 s^3 + 2.178 s^2 + 0.015 s + 0.01)} \tag{3.82}$$

Equation (3.79) shows that, at this flight condition, the characteristic motion of aircraft BRAVO is composed of phugoid mode, with damping ratio, ζ_{ph}, of 0.073 and frequency, ω_{ph}, of 0.0682 rad s^{-1}, and a short period mode with damping ratio, ζ_{sp}, of 0.557 and frequency, ω_{sp}, of 1.774 rad s^{-1}.

3.5 TRANSFER FUNCTIONS OBTAINED FROM SHORT PERIOD APPROXIMATION

3.5.1 Pitch Rate and Angle-of-attack Transfer Functions

The short period approximation consists of assuming that any variations, u, which arise in airspeed as a result of control surface deflection, atmospheric turbulence, or just aircraft motion, are so small that any terms in the equations of motion involving u are negligible. In other words, the approximation assumes that short period transients are of sufficiently short duration that U_0 remain essentially constant, i.e. $u = 0$. Thus, the equations of longitudinal motion may now be written as:

$$\dot{w} = Z_w w + U_0 q + Z_{\delta_E}\delta_E \tag{3.83}$$

$$\begin{aligned}\dot{q} &= M_w w + M_{\dot{w}}\dot{w} + M_q q + M_{\delta_E}\delta_E = (M_w + M_{\dot{w}}Z_w)w \\ &\quad + (M_q + U_0 M_{\dot{w}})q + (M_{\delta_E} + Z_{\delta_E}M_{\dot{w}})\delta_E\end{aligned} \tag{3.84}$$

If the state vector for short period motion is now defined as:

$$\mathbf{x} \triangleq \begin{bmatrix} w \\ q \end{bmatrix} \tag{3.85}$$

and the control vector, u, is taken as the elevator deflection, δ_E, then eqs (3.83) and (3.84) may be written as a state equation:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{3.86}$$

where:

$$A = \begin{bmatrix} Z_w & U_0 \\ (M_w + M_{\dot{w}}Z_w) & (M_q + U_0 M_{\dot{w}}) \end{bmatrix} \tag{3.87}$$

$$B = \begin{bmatrix} Z_{\delta_E} \\ (M_{\delta_E} + Z_{\delta_E}M_{\dot{w}}) \end{bmatrix} \tag{3.88}$$

$$\therefore \quad [sI - A] = \begin{bmatrix} (s - Z_w) & -U_0 \\ -(M_w + M_{\dot{w}}Z_w) & (s - [M_q + U_0 M_{\dot{w}}]) \end{bmatrix} \tag{3.89}$$

$$\begin{aligned}\Delta_{sp}(s) = \det[sI - A] &= s^2 - [Z_w + M_q + M_{\dot{w}}U_0]s + [Z_w M_q - U_0 M_w] \\ &= s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2\end{aligned} \tag{3.90}$$

where:

$$2\zeta_{sp}\omega_{sp} = -(Z_w + M_q + M_{\dot{w}}U_0) \tag{3.91}$$

$$\omega_{sp} = (Z_w M_q - U_0 M_w)^{1/2} \tag{3.92}$$

It is easy to show that:

$$\frac{w(s)}{\delta_E(s)} = \frac{(U_0 M_{\delta_E} - M_q Z_{\delta_E})\left\{1 + \dfrac{Z_{\delta_E}}{(U_0 M_{\delta_E} - M_q Z_{\delta_E})}s\right\}}{\Delta_{sp}(s)} = \frac{K_w(1 + sT_1)}{\Delta_{sp}(s)} \tag{3.93}$$

where:

$$K_w = (U_0 M_{\delta_E} - M_q Z_{\delta_E}) \tag{3.94}$$

$$T_1 = Z_{\delta_E}/K_w \tag{3.95}$$

Also:

$$\frac{q(s)}{\delta_E(s)} = \frac{(Z_{\delta_E} M_w - M_{\delta_E} Z_w)\left\{1 + \dfrac{M_{\delta_E} + Z_{\delta_E} M_{\dot{w}}}{(Z_{\delta_E} M_w - M_{\delta_E} Z_w)} s\right\}}{\Delta_{sp}(s)} = \frac{K_q(1 + sT_2)}{\Delta_{sp}(s)} \tag{3.96}$$

where:

$$K_q = (Z_{\delta_E} M_w - M_{\delta_E} Z_w) \tag{3.97}$$

$$T_2 = (M_{\delta_E} + Z_{\delta_E} M_{\dot{w}})/K_q \tag{3.98}$$

3.5.2 The Effect of Changes in Static Stability on Short Period Dynamics

When the steady forward speed is fixed, it is possible to increase the value of the short period damping ratio, ζ_{sp}, by augmenting (increasing) any or all of the stability derivatives: Z_w, $M_{\dot{w}}$ and M_q.

If $M_{\dot{w}}$ is augmented, T_2 is increased; the value of the short period frequency, ω_{sp}, is unchanged. If the value of M_{δ_E} is arranged to be equal to $Z_{\delta_E} M_{\dot{w}}$ it is possible for T_2 to be zero.

Augmenting the value of M_q causes an increase in the value of the damping ratio of the short period motion. The frequency of the short period mode is also increased by this change in the value of M_q. The value of T_1 is reduced, although the value of T_2 remains unchanged.

The damping ratio, ζ_{sp}, is also a direct function of M_w, the stability derivative whose value is related to the static stability. When the value of M_w approaches zero, the damping ratio of the short period increases, since the value of the natural frequency is reduced. If the aircraft is statically unstable, M_w is positive and if $U_0 M_w > M_q Z_w$ the aircraft will become dynamically unstable (see Section 3.2.3).

3.5.3 The Aircraft Time Constant

If the inequality (3.99) holds, i.e.:

$$Z_{\delta_E} M_w \ll M_{\delta_E} Z_w \tag{3.99}$$

and if:

$$Z_{\delta_E} M_{\dot{w}} \to 0 \tag{3.100}$$

then:

$$T_2 = M_{\delta_E} / M_{\delta_E} Z_w = -1/Z_w = T_A \tag{3.101}$$

T_2 is usually referred to as the aircraft time constant. How good the approximation is may be judged from Table 3.1 in which are quoted, for a wide

Table 3.1 Aircraft time constants

Parameter	*Aircraft type*						
	A-4D	*F-4C*	*Jaguar*	*Jetstar*	*DC-8*	*B-747*	*C-5a*
Z_w	− 0.307	− 0.452	− 0.6	− 1.01	− 0.63	− 0.512	− 0.634
T_2^{-1}	0.31	0.39	0.57	0.95	0.56	0.49	0.595

variety of aircraft operating at about the same flight condition, the values of Z_w and of the inverse of T_2 (determined from the full equations).

3.5.4 Flight Path Angle

There is a useful kinematic relationship which can be found by means of the short period approximation: to change the flight path angle, γ, of an aircraft it is customary to command a change in the pitch attitude, θ, of the aircraft. Since

$$\gamma = \theta - \alpha \tag{3.102}$$

$$\frac{\gamma(s)}{\theta(s)} = 1 - \frac{\alpha(s)}{\delta_E(s)} \cdot \frac{\delta_E(s)}{\theta(s)} \tag{3.103}$$

By means of eqs (3.93) and (3.96), and remembering that:

$$\alpha = w/U_0 \tag{3.104}$$

and

$$\dot{\theta} = q \tag{3.105}$$

it can be shown that:

$$\frac{\gamma(s)}{\theta(s)} = \frac{-Z_{\delta_E}s^2 + [U_0(M_{\delta_E} + Z_{\delta_E}M_{\dot{w}}) - (M_{\delta_E}U_0 - M_qZ_{\delta_E})s]}{U_0(M_{\delta_E} + Z_{\delta_E}M_{\dot{w}})s + U_0(M_wZ_{\delta_E} - Z_wM_{\delta_E})} + \frac{U_0(M_wZ_{\delta_E} - Z_wM_{\delta_E})}{U_0(M_{\delta_E} + Z_{\delta_E}M_{\dot{w}})s + U_0(M_wZ_{\delta_E} - Z_wM_{\delta_E})} \tag{3.106}$$

Generally Z_{δ_E} is negligible. Then:

$$\frac{\gamma(s)}{\theta(s)} \rightarrow \frac{-Z_wM_{\delta_E}U_0}{U_0(M_{\delta_E}s - Z_wM_{\delta_E})} = \frac{-Z_w}{(s - Z_w)} = \frac{1}{1 + sT_A}$$

where:

$$T_A = -Z_w^{-1} \tag{3.107}$$

as before. From eq. (3.107) it is easy to derive that:

$$\dot{\gamma} = \alpha/T_A \tag{3.108}$$

3.6 TRANSFER FUNCTIONS OBTAINED FROM PHUGOID APPROXIMATION

Lanchester (Sutton, 1949) studied the slow period motion of aircraft and noted that the phugoid motion consists of large oscillatory changes in speed u, height h, and pitch attitude θ. In that classic treatment, Lanchester took the value of the stability derivative M_u, i.e. the change in pitching moment due to changes in airspeed, to be negligible for all aircraft, i.e.:

$$M_u = 0 \tag{3.109}$$

However, for modern aircraft M_u is seldom zero and the total static stability moment of the aircraft becomes:

$$M_u u + M_w w = 0 \tag{3.110}$$

Since short period changes in q, for example, are not of interest the equations of motion can be written as:

$$\begin{aligned} \dot{u} &= X_u u + X_w w - g\theta + X_\delta \delta \\ \dot{w} &= Z_u u + Z_w w + U_0 q + Z_\delta \delta \\ 0 &= M_u u + M_w w + M_\delta \delta \end{aligned} \qquad (3.111)[5]$$

Hence, taking Laplace transforms:

$$\begin{aligned} su(s) - X_u u(s) - X_w w(s) + g\theta(s) &= X_\delta \delta(s) \\ sw(s) - Z_u u(s) - Z_w w(s) - U_0 s\theta(s) &= Z_\delta \delta(s) \\ - M_u u(s) - M_w w(s) &= M_\delta \delta(s) \end{aligned} \tag{3.112}$$

i.e.:

$$\begin{bmatrix} (s - X_u) & -X_w & g \\ -Z_u & (s - Z_w) & -U_0 s \\ -M_u & -M_w & 0 \end{bmatrix} \begin{bmatrix} u(s) \\ w(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} X_\delta \\ Z_\delta \\ M_\delta \end{bmatrix} \delta(s) \tag{3.113}$$

i.e.:

$$Q(s)x(s) = P(s)\delta(s) \tag{3.114}$$

hence:

$$x(s) = Q^{-1}(s)P(s)\delta(s) \tag{3.115}$$

$$\therefore \quad \frac{u(s)}{\delta(s)} = \frac{s[X_w U_0 M_\delta - gM_\delta - U_0 M_w X_\delta] + g[M_\delta Z_w - M_w Z_\delta]}{\Delta_{ph}(s)} \tag{3.116}$$

$$\frac{\theta(s)}{\delta(s)} = \frac{s^2 M_\delta + [M_u X_\delta + M_w Z_\delta - (X_u + Z_w)M_\delta]s}{\Delta_{ph}(s)} \tag{3.117}$$

$$+\frac{[(Z_uM_w - M_uZ_w)X_\delta + (M_uX_w - M_wX_u)Z_\delta] + (Z_wX_u - X_wZ_u)M_\delta}{\Delta_{ph}(s)}$$

where:

$$\Delta_{ph}(s) = -U_0M_w\left\{s^2 - \left[x_u + \frac{M_u(U_0X_w - g)}{U_0M_w}\right]s - \frac{g}{U_0}\left[Z_u - \frac{M_uZ_w}{M_w}\right]\right\} \tag{3.118}$$

From eq. (3.118):

$$\omega_{ph}^2 = \frac{-g}{U_0}\left[Z_u - \frac{M_uZ_w}{M_w}\right] \tag{3.119}$$

$$2\zeta_{ph}\omega_{ph} = -\left[X_u + \frac{M_u(U_0X_w - g)}{U_0M_w}\right] \tag{3.120}$$

If M_u is sufficiently negative the result is that ω_{ph}^2 becomes negative: that unstable mode is called the divergent tuck mode (see Section 3.2.2).

If Lanchester's classical approximation is invoked, i.e. $M_u = 0$, then:

$$2\zeta_{ph}\omega_{ph} = -X_u \tag{3.121}$$

and

$$\omega_{ph}^2 = -gZ_u/U_0 \tag{3.122}$$

The stability derivative, Z_u, can be shown to be:

$$Z_u \simeq \frac{-\rho SU_0}{m}C_L \tag{3.123}$$

but the lift coefficient, C_L, can be shown to be (in steady, straight and level flight):

$$C_L = \frac{\text{weight}}{\bar{q}S} = \frac{2mg}{\rho U_0^2S} \tag{3.124}$$

$$\therefore \quad Z_u = -2g/U_0 \tag{3.125}$$

$$\therefore \quad \omega_{ph} \simeq \sqrt{2}\frac{g}{U_0} \tag{3.126}$$

Based on the assumption that the stability derivative, M_u, had a value of zero, the resulting approximation, the classical phugoid approximation, was called the two degrees of freedom phugoid approximation, i.e.:

$$\begin{aligned}(s - X_u)\,u(s) + g\theta(s) &= 0\\ -Z_uu(s) - U_0s\theta(s) &= Z_\delta\delta(s)\end{aligned} \tag{3.127}$$

Therefore the characteristic equation is

Table 3.2 Comparison of phugoid parameters

Aircraft type	U_0 *(m s^{-1})*	Z_u		ω_{ph}		$\omega_{ph} = -Z_u/\sqrt{2}$	M_u
		Actual	*Calculated* eq. (3.125)	*Actual*	*Calculated* eq. (3.126)		
DC-3	45	− 0.476	− 0.474	0.301	0.33	0.337	0.0
F-89	210	− 0.0955	− 0.0976	0.063	0.069	0.0678	0.0
DC-8	285	− 0.0735	− 0.076	(− 0.0016)	0.053	0.0527	− 0.00254

$$s^2 - X_u s - (gZ_u/U_0) = 0 \tag{3.128}$$

For modern aircraft, the three degrees of freedom approximation represented by eq. (3.114) is preferred. Table 3.2 illustrates the character of the approximations.

It can be seen from the table that the classical, and even the three degrees of freedom, approximation is unacceptable in the case of the DC-8 where a divergent tuck mode exists.

In the classical approximation, the assumption that the value of M_u is zero corresponds to an assumption that the coefficient of drag due to changes in forward speed, C_{D_u}, is also zero.

Now,

$$\zeta_{ph} = -\frac{X_u}{2\omega_{ph}} = -\frac{X_u U_0}{2\sqrt{(2g)}} \tag{3.129}$$

However,

$$X_u = \frac{\bar{q}S}{mU_0}(C_{D_u} + 2C_D) \simeq \frac{2\bar{q}S}{mU_0} C_D \tag{3.130}$$

From eq. (3.124):

$$C_L = \frac{2mg}{\rho U_0^2 S} \tag{3.124}$$

$$\therefore \quad X_u = \frac{2g}{U_0}\frac{C_D}{C_L} \tag{3.131}$$

Hence,

$$\zeta_{ph} = -\left(\frac{2g}{U_0}\right)\left(\frac{C_D}{C_L}\right)\frac{U_0}{2\sqrt{(2g)}} = \frac{1}{\sqrt{2}}\frac{C_D}{C_L} = \frac{1}{\sqrt{2}(L/D)} \tag{3.132}$$

where L/D is the lift/drag ratio of the aircraft. For example, at $U_0 = 210\ \mathrm{m\ s^{-1}}$ the F-89 has a lift/drag ratio of 12.0, therefore:

$$\zeta_{ph} = 1/(\sqrt{2} \times 12) \simeq 0.06$$

$$\omega_{ph} = \sqrt{2} \times 9.81/210 \simeq 0.0661\ \mathrm{rad\ s^{-1}}$$

3.7 LATERAL STABILITY

The characteristic polynomial of lateral motion, $\det[\lambda I - A]$, is of fifth degree, i.e. it is a quintic of the form: $\lambda^5 + d_1\lambda^4 + d_2\lambda^3 + d_3\lambda^2 + d_4\lambda$. This 'stability quintic' can usually be factorized into the following form: $\lambda(\lambda + e)(\lambda + f)(\lambda^2 + 2\zeta_D\omega_D\lambda + \omega_D^2)$. The simple term in λ corresponds to the heading (directional) mode. Because $\lambda = 0$ is a root of the characteristic equation, once an aircraft's heading has been changed, by whatever agency, there is no natural tendency for the aircraft to be restored to its equilibrium heading. An aircraft has neutral heading stability and it will remain at its perturbed heading until some corrective control action is taken. The term $(\lambda + e)$ corresponds to the spiral convergence/divergence mode, which is usually a very slow motion corresponding to a long term tendency either to maintain the wings level or to 'roll off' in a divergent spiral. The term $(\lambda + f)$ corresponds to the rolling subsidence mode; the quadratic term represents the 'dutch roll' motion for which the value of damping ratio, ζ_D, is usually small, so that 'dutch' rolling motion is oscillatory.

When the dihedral on the wing is great, and roll damping (L'_p) is low, the roll and spiral modes can couple and become a single roll/spiral oscillation (often referred to as the 'lateral phugoid' mode). If such aircraft have also a very lightly damped 'dutch roll' mode, then these aircraft have poor handling qualities and are difficult to fly.

3.8 TRANSFER FUNCTIONS RELATED TO LATERAL MOTION

3.8.1 State and Output Equations

By following the method used in Section 3.4 a number of important transfer functions relating to lateral motion can be found. However, in this case there are, even for conventional aircraft, two control surfaces, the aileron and the rudder, which are used simultaneously in certain phases of flight, such as final approach. When two inputs act simultaneously, then the use of transfer functions is less exact, since they are strictly single-input, single-output functions.

If the state vector for straight and level lateral motion is taken as that defined in eq. (2.151), namely:

$$\mathbf{x} = \begin{bmatrix} \beta \\ p \\ r \\ \phi \\ \psi \end{bmatrix} \tag{2.151}$$

and the control vector is defined as

$$\mathbf{u} = \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix} \tag{2.143}$$

the corresponding coefficient and driving matrices are given in eqs (2.152) and (2.153) as:

$$A = \begin{bmatrix} Y_v & 0 & -1 & \frac{g}{U_0} & 0 \\ L'_\beta & L'_p & L'_r & 0 & 0 \\ N'_\beta & N'_p & N'_r & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{2.152}$$

$$B = \begin{bmatrix} 0 & Y^*_{\delta_R} \\ L'_{\delta_A} & L'_{\delta_R} \\ N'_{\delta_A} & N'_{\delta_R} \\ 0 & 0 \end{bmatrix} \tag{2.153}$$

Assuming that no acceleration term, such as $a_{y_{cg}}$, is defined as an output variable and that the output will be taken as a single state variable, then:

$$y = C\mathbf{x} \tag{3.133}$$

If:

$$y \triangleq \beta \tag{3.134}$$

then:

$$C_\beta = [1 \ 0 \ 0 \ 0 \ 0] \tag{3.135a}$$

Similarly, the following output matrices can be defined:

$$\begin{aligned} C_p &= [0 \ 1 \ 0 \ 0 \ 0] \qquad C_\phi = [0 \ 0 \ 0 \ 1 \ 0] \\ C_r &= [0 \ 0 \ 1 \ 0 \ 0] \qquad C_\psi = [0 \ 0 \ 0 \ 0 \ 1] \end{aligned} \tag{3.135b}$$

If the transfer function being evaluated depends upon the aileron deflection, δ_A, the first column of the driving matrix, B, is used; the second column of B is used when the control input is the rudder deflection δ_R. Consequently, the development will proceed using δ as a control input; the appropriate subscript A or R should be added when the input is particular, and the corresponding values of the control stability derivatives Y^*_δ, L'_δ, and N'_δ should be used.

3.8.2 Transfer Functions in Terms of Stability Derivatives

From eq. (2.152) it is evident that the characteristic polynomial will be a quintic (i.e. of fifth degree) since:

$$\begin{aligned}\det[sI - A] &= s^5 - (L'_p + N'_r + Y_v)s^4 \\ &\quad + (L'_p N'_r - L'_r N'_p + Y_v L'_p + Y_v N'_r + N'_\beta)s^3 \\ &\quad + \left(L'_\beta N'_p - L'_p N'_\beta - \frac{g}{U_0} L'_\beta - Y_v L'_p N'_r + Y_v L'_r N'_p\right)s^2 \\ &\quad + \left(\frac{g}{U_0}[N'_r L'_\beta - L'_r N'_\beta]\right)s \\ &= s\,\Delta_{\text{lat}}(s) \end{aligned} \tag{3.136}$$

where

$$\Delta_{\text{lat}}(s) = s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4 \tag{3.137}$$

$$d_1 = -(L'_p + N'_r + Y_v) \tag{3.138}$$

$$d_2 = (L'_p N'_r - L'_r N'_p + Y_v L'_p + Y_v N'_r + N'_\beta) \tag{3.139}$$

$$d_3 = \left(L'_\beta N'_p - L'_p N'_\beta - \frac{g}{U_0} L'_\beta - Y_v L'_p N'_r + Y_v L'_r N'_p\right) \tag{3.140}$$

$$d_4 = \frac{g}{U_0}[N'_r L'_\beta - L'_r N'_\beta] \tag{3.141}$$

The adjoint of $[sI - A]$ takes the form:

$$\text{adj}[sI - A] \triangleq \begin{bmatrix} n_{11}(s) & n_{12}(s) & n_{13}(s) & n_{14}(s) & n_{15}(s) \\ n_{21}(s) & n_{22}(s) & n_{23}(s) & n_{24}(s) & n_{25}(s) \\ n_{31}(s) & n_{32}(s) & n_{33}(s) & n_{34}(s) & n_{35}(s) \\ n_{41}(s) & n_{42}(s) & n_{43}(s) & n_{44}(s) & n_{45}(s) \\ n_{51}(s) & n_{52}(s) & n_{53}(s) & n_{54}(s) & n_{55}(s) \end{bmatrix} \tag{3.142}$$

where $n_{ij}(s)$ is a cofactor of $[sI - A]$. The cofactors are:

$$n_{11}(s) = s^2\{s^2 - [L'_p + N'_r]s + [N'_r L'_p - L'_r N'_p]\} \tag{3.143}$$

$$n_{12}(s) = -s\left\{\left(N'_p - \frac{g}{U_0}\right]s + \frac{N'_r g}{U_0}\right\} \tag{3.144}$$

$$n_{13}(s) = -s\left\{s^2 - L'_p s - \frac{L'_r}{U_0}\right\} \tag{3.145}$$

$$n_{21}(s) = s^2\{sL'_\beta + [N'_\beta L'_r - L'_\beta N'_r]\} \tag{3.146}$$

$$n_{22}(s) = s^2\{s^2 - [Y_v + N'_r]s + [Y_v N'_r + N'_\beta]\} \tag{3.147}$$

$$n_{23}(s) = s^2\{sL'_r - [Y_v L'_r + L'_\beta]\} \tag{3.148}$$

$$n_{31}(s) = s^2\{sN'_\beta + [L'_\beta N'_p - L'_p N'_\beta]\} \tag{3.149}$$

$$n_{32}(s) = s\left\{s^2 N'_p - Y_v N'_p s + \frac{N'_\beta g}{U_0}\right\} \tag{3.150}$$

$$n_{33}(s) = s\left\{s^3 - [Y_v + L'_p]s^2 + Y_v L'_p s - \frac{g}{U_0} L'_p\right\} \tag{3.151}$$

$$n_{41}(s) = s\{sL'_\beta + [N'_\beta L'_r - L'_\beta N'_r]\} \tag{3.152}$$

$$n_{42}(s) = s\{s^2 - [Y_v + N'_r]s + [Y_v N'_r + N'_\beta]\} \tag{3.153}$$

$$n_{43}(s) = s\{sL'_r - [Y_v L'_r + L'_\beta]\} \tag{3.154}$$

$$n_{51}(s) = s^3\{sN'_\beta + [L'_\beta N'_p - N'_\beta L'_p]\} \tag{3.155}$$

$$n_{52}(s) = s^2\left\{s^2 N'_p - Y_v N'_p s + \frac{N'_\beta g}{U_0}\right\} \tag{3.156}$$

$$n_{53}(s) = s^2\left\{s^3 - [Y_v + L'_p]s^2 + Y_v L'_p s - \frac{g}{U_0} L'_\beta\right\} \tag{3.157}$$

Those cofactors not listed above are zero. Obviously,

$$\frac{\beta(s)}{\delta(s)} = C_\beta[sI - A]^{-1}B = \frac{[n_{11}(s)\ n_{12}(s)\ n_{13}(s)\ n_{14}(s)\ n_{15}(s)]}{s\Delta_{lat}(s)}B$$

$$= \frac{n_{11}(s)\,Y^*_\delta + n_{12}(s)\,L'_\delta + n_{13}(s)\,N'_\delta}{s\Delta_{lat}(s)} \tag{3.158}$$

$$\therefore \quad \frac{\beta(s)}{\delta(s)} = \frac{Y^*_\delta s^3 - [(L'_p + N'_r)Y^*_\delta + N'_\delta]s^2}{\Delta_{lat}(s)}$$

$$+ \frac{\left[(N'_r L'_p - L'_r N'_p)Y^*_\delta + (L'_p N'_\delta - N'_p L'_\delta) + \dfrac{gL'_\delta}{U_0}\right]s}{\Delta_{lat}(s)} \tag{3.159}$$

$$+ \frac{\dfrac{g}{U_0}(N'_\delta L'_r - N'_r L'_\delta)}{\Delta_{lat}(s)}$$

$$\frac{p(s)}{\delta(s)} = s\left\{\frac{s^2 L'_\delta + [(L'_\beta Y^*_\delta - L'_\delta(N'_r + Y_v) + N'_\delta L'_r]s}{\Delta_{lat}(s)}\right.$$

$$\left. + \frac{Y^*_\delta(L'_r N'_\beta - N'_r L'_r) + L'_\delta(Y_v N'_r + N'_\beta) - N'_\delta(L'_\beta + Y_v L'_r)}{\Delta_{lat}(s)}\right\}$$

$$\therefore \quad \frac{s\phi(s)}{\delta(s)} \triangleq \frac{N^p_\delta(s)}{\Delta_{lat}(s)} \tag{3.160}$$

$$\frac{r(s)}{\delta(s)} = \frac{s\psi(s)}{\delta(s)}$$

$$= \frac{N'_\delta s^3 + [Y^*_\delta N'_\beta + L'_\delta N'_p - N'_\delta(Y_v + L'_p)]s^2}{\Delta_{lat}(s)} \quad (3.161)$$

$$+ \frac{[Y^*_\delta(L'_\beta N'_p - N'_\beta L'_p) - L'_\delta Y_v N'_p + N'_\delta Y_v L'_p]s}{\Delta_{lat}(s)}$$

$$+ \frac{\frac{g}{U_0}[L'_\delta N'_\beta - N'_\delta L'_\beta)}{\Delta_{lat}(s)}$$

3.8.3 Lateral Acceleration as an Output Variable

If the transfer function relating the lateral acceleration at the aircraft's c.g. to some control input δ is required, it may be obtained by noting that, from eq. (2.158),

$$a_{y_{cg}} = C_{a_y}\mathbf{x} + D\mathbf{u} \triangleq y \quad (3.162)$$

$$C_{a_y} = [Y_v \ 0 \ 0 \ 0 \ 0] \quad (3.163)$$

$$D = [0 \ Y^*_{\delta_R}] \quad (3.164)$$

3.8.4 Some Representative Transfer Functions

Taking the large passenger jet aircraft CHARLIE in Appendix B, for flight condition 4, the following transfer functions can be evaluated:

$$\frac{\beta(s)}{\delta_R(s)} = \frac{0.012(s - 0.027)(s + 0.52)(s + 40.1)}{(s - 0.012)(s + 0.562)(s + 0.091s + 0.656)} \quad (3.165)$$

$$\frac{r(s)}{\delta_R(s)} = \frac{-0.48(s + 0.587)(s^2 - 0.066s + 0.059)}{(s - 0.012)(s + 0.562)(s^2 + 0.091s + 0.656)} \quad (3.166)$$

$$\frac{p(s)}{\delta_A(s)} = \frac{0.14(s^2 - 0.2s + 0.668)}{(s - 0.012)(s + 0.562)(s^2 + 0.091s + 0.656)} \quad (3.167)$$

$$\frac{a_{y_{cg}}(s)}{\delta_R(s)} = \frac{0.012(s - 0.027)(s - 31.6)(s + 17.744)(s + 0.52)}{(s - 0.012)(s + 0.562)(s^2 + 0.091s + 0.656)} \quad (3.168)$$

3.8.5 Some Transfer Function Approximations

In every transfer function, except eq. (3.167), the dutch roll mode is a major component of the weighting function of the aircraft, i.e. its response to an

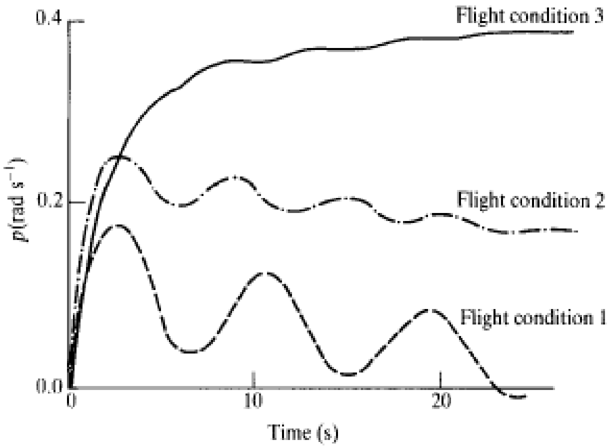


Figure 3.4 Roll rate response for CHARLIE.

impulse. For the transfer function, $p(s)/\delta_A(s)$, the quadratic numerator term very nearly cancels the quadratic term in the denominator. If that cancellation were exact, no dutch roll motion would be evident in the rolling motion of the aircraft; however, there is usually a small amount evident (see Figure 3.4). In every transfer function above, except the rolling motion transfer function eq. (3.167), a first order numerator term almost exactly cancels the term $(s + 0.56)$ in the denominator; this term corresponds to the rolling subsidence mode. The transfer function relating the lateral acceleration at the c.g. to a rudder deflection $a_{y_{cg}}(s)/\delta_R(s)$ approximates very closely to a constant value of -33.0, because all the numerator terms very nearly cancel the corresponding denominator terms. Inspection of the transfer function eq. (3.165) shows that a much simpler, approximate form might be used, namely:

$$\frac{\beta(s)}{\delta_R(s)} = \frac{0.012(s + 40.1)}{(s^2 + 0.091s + 0.656)}$$

$$= \frac{0.481(1 + 0.025s)}{(s^2 + 0.091s + 0.656)} \tag{3.169}$$

The time constant of the numerator term, $0.025s$, is very short and can be ignored, so that the approximation may be taken as:

$$\frac{\beta(s)}{\delta_R(s)} = \frac{0.48}{(s^2 + 0.091s + 0.656)} \tag{3.170}$$

The primary response to aileron deflection is in roll rate and the evidence of any dutch roll motion excited by an aileron deflection is principally found in sideslip β and yaw rate r. In the spiral motion of an aircraft, rolling and yawing motion are predominant and, although the mode is usually unstable, the motion is very nearly co-ordinated. Sideslip is almost non-existent in the spiral mode, and the motion which occurs is a co-ordinated bank turn defined by:

$$a_{y_{cg}} = \ddot{y} = U_0(\dot{\beta} + r) = U_0(v + r) = 0 \tag{3.171}$$

$$\dot{y} = U_0(\beta + \psi) = 0 \tag{3.172}$$

where y denotes lateral displacement.

3.9 THREE DEGREES OF FREEDOM APPROXIMATIONS

3.9.1 Dutch Roll Approximation

From a consideration of the appropriate cancellation of terms in transfer functions, it appears likely that there are some useful approximations which can lead to simpler transfer functions of acceptable accuracy, which still represent the functional relationship between the motion variable of the aircraft and the control deflection which caused it. The first of these approximations is the three degrees of freedom approximation which is arrived at by taking the equations of motion for straight and level flight given by eq. (2.85), and neglecting a few insignificant terms. Thus, the following terms are small for small perturbation motion and flight at moderate and higher speeds, and are assumed to be zero: the term due to gravity, $g\phi/U_0$; rolling acceleration as a result of yaw rate, $L'_r r$; yawing acceleration as a result of roll rate, $N'_p p$. Therefore, the equations of motion may now be written as:

$$\begin{aligned} \dot{\beta} &= Y_v\beta - r + Y^*_\delta\delta \\ \dot{p} &= L'_\beta\beta + L'_p p + L'_\delta\delta \\ \dot{r} &= N'_\beta\beta + N'_r r + N'_\delta\delta \end{aligned} \tag{3.173}$$

i.e.

$$\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} Y_v & 0 & -1 \\ L'_\beta & L'_p & 0 \\ N'_\beta & 0 & N'_r \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \end{bmatrix} + \begin{bmatrix} Y^*_\delta \\ L'_\delta \\ N'_\delta \end{bmatrix} \delta \tag{3.174}$$

This is referred to as the dutch roll approximation.

From eq. (3.174) it is easy to show that:

$$\begin{aligned} \Delta_{lat}(s) = s^3 - [Y_v + L'_p + N'_r]s^2 + [Y_vL'_p + Y_vN'_p + L'_pN'_r + N'_\beta]s \\ - L'_p[N'_\beta + N'_rY_v] \end{aligned} \tag{3.175}$$

and that, for example,

$$\frac{\beta(s)}{\delta_R(s)} = \frac{Y^*_\delta s^2 - [(L'_p + N'_r)Y^*_\delta + N'_\delta]s + [L'_pN'_rY^*_\delta + L'_pN'_\delta]}{\Delta_{lat}(s)} \tag{3.176}$$

$$\frac{p(s)}{\delta_A(s)} = \frac{L'_\delta s^2 + [Y^*_\delta L'_\beta - Y_v L'_\delta - N'_r L'_\delta]s}{\Delta_{lat}(s)} + \frac{[Y_v N'_r L'_\delta + N'_\beta L'_\delta - L'_\beta N'_\delta - Y^*_\delta L'_\beta N'_r]}{\Delta_{lat}(s)} \quad (3.177)$$

3.9.2 An Example of Dutch Roll Approximation

For aircraft CHARLIE at flight condition 4,

$$\frac{p(s)}{\delta_A(s)} = \frac{0.14(s^2 + 0.193s + 0.673)}{(s + 0.506)(s^2 + 0.135s + 0.63)} \quad (3.178)$$

By cancelling the quadratic terms of the numerator and denominator the resulting transfer function becomes:

$$\frac{p(s)}{\delta_A(s)} = \frac{0.14}{(s + 0.506)} \quad (3.179)$$

Inspection of eq. (3.167), with appropriate cancellations, will indicate how closely the results correspond. The transfer function for the same aircraft and flight condition is easily determined:

$$\frac{\beta(s)}{\delta_R(s)} = \frac{0.012(s + 0.48)(s + 40.1406)}{(s + 0.506)(s^2 + 0.135s + 0.68)} \simeq \frac{0.481(1 + 0.025s)}{(s^2 + 0.135s + 0.68)} \quad (3.180)$$

Since the time constant of the numerator term is negligibly small, the approximate transfer function is given by:

$$\frac{\beta(s)}{\delta_R(s)} = \frac{0.48}{(s^2 + 0.14s + 0.68)} \quad (3.181)$$

which should be compared with eq. (3.170): note how close the transfer functions are.

3.9.3 Spiral and Roll Subsidence Approximations

The approximations are founded on the observation that, for both spiral and roll modes, the corresponding sideslip motion is small and that, for the spiral mode, the term $\dot{\beta}$ is negligible with respect to the remaining terms in the equation for side force. Consequently, eq. (2.85) can be rewritten as:

$$0 = \frac{g}{U_0}\phi - r + Y^*_\delta \delta + Y_v \beta$$

$$\dot{p} = L'_\beta \beta + L'_p p + L'_r r + L'_\delta \delta \quad (3.182)$$

$$\dot{r} = N'_\beta \beta + N'_p p + N'_r r + N'_\delta \delta$$
$$\dot{\phi} = p$$

Thus, when $\beta = 0$, the equations of motion reduce to the following set:

$$\begin{aligned} \dot{p} &= L'_p p + L'_r r + L'_{\delta_R} \delta_R \\ \dot{r} &= N'_p p + N'_r r + N'_{\delta_R} \delta_R \\ \dot{\phi} &= p \end{aligned} \tag{3.183}$$

i.e. if:

$$\mathbf{x} \triangleq \begin{bmatrix} p \\ r \\ \phi \end{bmatrix} \tag{3.184a}$$

$$u \triangleq \delta_R \tag{3.184b}$$

then:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{3.184c}$$

when:

$$A \triangleq \begin{bmatrix} L'_p & L'_r & 0 \\ N'_p & N_r & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{3.185a}$$

$$B \triangleq \begin{bmatrix} L'_{\delta_R} \\ N'_{\delta_R} \\ 0 \end{bmatrix} \tag{3.185b}$$

To find the sideslip angle, β, which results from a rudder deflection, it is necessary to define β as an output variable, y, i.e.

$$\begin{aligned} y \triangleq \beta &= \frac{1}{Y_v} r - \frac{g}{U_0 Y_v} \phi - \frac{Y^*_{\delta_R}}{Y_v} \delta_R \\ &= \left[0 \ \frac{1}{Y_v} \ - \frac{g}{U_0 Y_v} \right] \mathbf{x} + \left[- \frac{Y^*_{\delta_R}}{Y_v} \right] u \\ &= C_\beta \mathbf{x} + D_\beta u \end{aligned} \tag{3.186}$$

For CHARLIE-4

$$A = \begin{bmatrix} -0.47 & 0.39 & 0.0 \\ -0.032 & -0.115 & 0.0 \\ 1.0 & 0.0 & 0.0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.15 \\ -0.48 \\ 0.0 \end{bmatrix}$$

$$C_\beta = [0.0 \ -17.857 \ 0.7]$$

$$D_\beta = [0.2143]$$

It can easily be shown that the following transfer functions are obtained using this three degree of freedom approximation, namely:

$$\frac{\beta(s)}{\delta_R(s)} = \frac{0.2143(s - 0.026)(s + 0.52)(s + 40.1)}{s(s + 0.155)(s + 0.43)} \tag{3.187}$$

$$\simeq \frac{0.2143(s + 40.1)}{(s + 0.155)}$$

$$\frac{p(s)}{\delta_R(s)} = \frac{0.15(s - 1.133)}{(s + 0.155)(s + 0.43)} \tag{3.188}$$

$$\frac{r(s)}{\delta_R(s)} = \frac{-0.48(s + 0.48)}{(s + 0.43)(s + 0.155)} \simeq \frac{-0.48}{(s + 0.155)} \tag{3.189}$$

It is evident from these transfer functions that the dutch roll mode is absent from this characterization, which is really unacceptable. Consequently, the approximation is rarely used.

3.10 TWO DEGREES OF FREEDOM APPROXIMATION

If it is assumed that the bank angle motion is negligible then the sum of the rolling moments is zero at all times; consequently, the roll equation is eliminated along with the bank angle perturbations. Thus:

$$\dot\beta = Y_v\beta - r + Y_\delta^*\delta \qquad \dot r = N'_\beta\beta + N'_r r + N'_\delta\delta \tag{3.190}$$

$$\begin{bmatrix} \dot\beta \\ \dot r \end{bmatrix} = \begin{bmatrix} Y_v & -1 \\ N'_\beta & N'_r \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} Y_\delta^* \\ N'_\delta \end{bmatrix} \delta \tag{3.191}$$

$$\therefore \quad \Delta_{lat}(s) = s^2 - (Y_v + N'_r)s + (N'_\beta + Y_v N'_r) \tag{3.192}$$

and

$$\frac{\beta(s)}{\delta_R(s)} = \frac{(s - N'_r)Y_\delta^* - N'_\delta}{\Delta_{lat}(s)} \tag{3.193}$$

$$\frac{r(s)}{\delta_R(s)} = \frac{(s - Y_v)N'_\delta + Y_\delta^* N'_\beta}{\Delta_{lat}(s)} \tag{3.194}$$

For aircraft CHARLIE, at flight condition 4:

$$\frac{\beta(s)}{\delta_R(s)} = \frac{0.012(s + 40.115)}{(s^2 + 0.173s + 0.61)} \tag{3.195}$$

$$\frac{r(s)}{\delta_R(s)} = \frac{-0.48(s + 0.4)}{(s^2 + 0.173s + 0.061)} \tag{3.196}$$

The approximation (3.195) is reasonably close to that obtained as eq. (3.169), although the damping ratio is about twice the proper value. Nevertheless this approximation (3.193) is used frequently in AFCS work.

3.11 SINGLE DEGREE OF FREEDOM APPROXIMATION

In this approximation only rolling motion is assumed to occur as a result of an aileron deflection, i.e.

$$\dot{p} = L'_p p + L'_{\delta_A}\delta_A \tag{3.197}$$

i.e.:

$$(s - L'_p)p(s) = L'_{\delta_A}\delta_A(s) \tag{3.198}$$

$$\therefore \quad \frac{p(s)}{\delta_A(s)} = \frac{L'_{\delta_A}}{(s - L'_p)} \tag{3.199}$$

For aircraft CHARLIE, at flight condition 4:

$$\frac{p(s)}{\delta_A(s)} = \frac{0.14}{(s + 0.47)} \tag{3.200}$$

If the corresponding numerator and denominator terms in eq. (3.167) are cancelled, the result is:

$$\frac{p(s)}{\delta_A(s)} = \frac{0.14}{(s + 0.56)} \tag{3.201}$$

which is very close to eq. (3.200). For bank angle control systems, the single degree of freedom approximation is frequently used as a first approximation.

3.12 STATE EQUATION FORMULATION TO EMPHASIZE LATERAL/DIRECTIONAL EFFECTS

If the state vector for lateral motion is defined thus:

$$\mathbf{x} = \begin{bmatrix} r \\ \beta \\ p \\ \phi \end{bmatrix} \tag{3.202}$$

and the control vector, $\mathbf{u}$, is defined as:

$$\mathbf{u} = \begin{bmatrix} \delta_R \\ \\ \delta_A \end{bmatrix} \tag{3.203}$$

then:

$$A = \begin{bmatrix} N'_r & N'_\beta & N'_p & 0 \\ -1 & Y_v & 0 & \dfrac{g}{U_0} \\ L'_r & L'_\beta & L'_p & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.204}$$

$$B = \begin{bmatrix} N'_{\delta_R} & N'_{\delta_A} \\ Y^*_{\delta_R} & Y^*_{\delta_A} \\ L'_{\delta_R} & L'_{\delta_A} \\ 0 & 0 \end{bmatrix} \tag{3.205}$$

By choosing the state and control vectors in this fashion, A can be partitioned as follows:

$$\left[\begin{array}{c|c} \textit{Directional effects} & \textit{Lateral/directional coupling} \\ \hline \textit{Directional/lateral coupling} & \textit{Lateral effects} \end{array}\right] \tag{3.206}$$

or, more compactly:

$$A = \left[\begin{array}{c|c} A_D & A_D^L \\ \hline A_L^D & A_L \end{array}\right] \tag{3.207}$$

In a similar way:

$$B = \left[\begin{array}{c|c} B_D & B_D^L \\ \hline B_L^D & B_L \end{array}\right] \tag{3.208}$$

The strength of the lateral/directional coupling depends upon the relative magnitude of the 'off-diagonal' blocks.

In A, the coupling effects are 'stability' effects, while the coupling effects in B represent control effects. They are quite separate phenomena. But coupling effects in A almost always lead to coupled control response whether or not there are any explicit coupling effects in B. Control coupling can affect stability only when there is external feedback as a result of a pilot's action or of the AFCS.

If the off-diagonal blocks are negligible then dutch roll motion is approximated by the directional equation:

$$\begin{bmatrix} \dot{r} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} N'_r & N'_\beta \\ -1 & Y_v \end{bmatrix} \begin{bmatrix} r \\ \beta \end{bmatrix} + \begin{bmatrix} N'_{\delta_R} & N'_{\delta_A} \\ Y^*_{\beta_R} & Y^*_{\delta_A} \end{bmatrix} \begin{bmatrix} \delta_R \\ \delta_A \end{bmatrix} \tag{3.209}$$

The lateral equation is given by:

$$\begin{bmatrix} \dot{p} \\ \phi \end{bmatrix} = \begin{bmatrix} L'_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ \phi \end{bmatrix} + \begin{bmatrix} L'_{\delta_A} \\ 0 \end{bmatrix} \delta_A \tag{3.120}$$

As always, the stability of the respective motions is governed by the roots of the characteristic equations. For the dutch roll motion it is easy to show that:

$$\omega_D = (N'_\beta + N'_r Y_v)^{1/2} \tag{3.211}$$

$$\zeta_D = -\frac{(N'_r + Y_v)}{2\omega_D} \tag{3.212}$$

For the lateral motion the characteristic equation is given by:

$$s(s - L'_p) = 0 \tag{3.213}$$

The time constant of the roll mode (which is described by the single degree of freedom approximation) is $-(L'_p)^{-1}$. The other mode – the spiral mode – is neutrally stable since the remaining eigenvalue associated with eq. (3.213) is 0.0. Effectively, this approximation assumes that $L'_\beta \rightarrow 0$ (i.e. the dihedral effect of the wing is small) and U_0 is large. For further discussion, the reader should refer to Stengel (1980).

3.13 CONCLUSIONS

There are many ways of representing the dynamics of an aircraft. Which form to choose depends principally upon the task being considered. Where only a single control input or a single source of disturbance is being considered, it is natural to use the transfer function approach: the relationship between the output and the input is unique. The state equation is not a unique representation of the aircraft dynamics, but depends upon the definition of the state and control and disturbance vectors. Nevertheless, even for cases where the designer can be certain that only a single forcing function applies, there is great merit in using state space methods since they afford information about the response of all the state variables to that single input, and not just about response of a single output.

More and more, modern AFCS problems are multivariable in their nature; state space methods are now the natural tools for design and analysis of such dynamic systems.

3.15 NOTES

1. A zero real part corresponds to a mode having simple harmonic motion, which, for practical flight situations, is considered to be unstable.
2. By computer, using NAG library routines (from NAG, Mayfield House, Oxford, England), or the routine available in the EISPACK package (Garbow *et al.*, 1977) or the EIG function in CTRL-C (Systems Control Technology, Inc., 1986).
3. It is assumed here, again, that any forces, which may arise owing to the thrust lines not coinciding with the aircraft axes, are negligible and may be ignored.
4. In aeronautics, volume is the product of the area of a flying surface and the distance of that surface from the c.g. of the aircraft measured to $0.25\,\bar{c}$ of the surface.
5. δ is used here to indicate any control surface deflection. To be specific an appropriate subscript is used.

3.16 REFERENCES

BARNETT, S. and C. STOREY. 1966. Insensitivity of optimal linear control systems to persistent changes in parameters. *Int. J. Control.*, 4(2): 179–84.

FADDEEVA, V.N. 1959. Computational Methods of Linear Algebra. New York: Dover Press.

GARBOW, B.S., J.M. BOYLE, J.J. DONGARRA and C.B. MOLER. 1977. *Matrix Eigensystem Routines – EISPACK Guide Extensions.* Springer-Verlag Lecture Notes in Computer Science, Vol 6.

LARIMER, S.J. 1978. TOTAL. MSc thesis, AFIT/ENE, WPAFB, Dayton, Ohio.

STENGEL, R.F. 1980. Some effects of parameter variations on the lateral–directional stability of aircraft. *J. Guid. and Cont.* 3(2): 124–31.

SUTTON, O.G. 1949. *The Science of Flight.* Harmondsworth: Penguin.

Systems Control Technology, Inc., 1986. *Ctrl-C Users Guide: Version 4.0.* SCT, Inc., Palo Alto, California.

4

The Dynamic Effects of Structural Flexibility Upon the Motion of an Aircraft

4.1 INTRODUCTION

The current design and mission requirements for military and commercial transport aircraft are such that the resulting configurations of such vehicles have required the use of thin lifting surfaces, long and slender fuselages, low mass fraction structures, high stress design levels, and low dynamic load factors. In turn, those features have resulted in aircraft which are structurally light and flexible. Such aircraft can develop large values of displacement and acceleration as a result of structural deflection, in addition to those components of displacement and acceleration which arise owing to the rigid body motion of the aircraft. Such structural deflections may occur as a result of aircraft manoeuvres which have been commanded by a pilot, or as a result of the aircraft's passage through turbulent air. Aircraft motion of this kind can result in a reduction of the structural life of the airframe because of the large dynamic loads and the consequent high levels of stress. The amplitude of the aircraft's response, caused by gust-induced structural flexibility, depends upon either the amount of energy transferred from the gust disturbance to the structural bending modes or, if any energy is absorbed from the gust, the dissipation of that energy by some form of damping. When the amplitude of the response of the elastic motion is such that it compares with that of the rigid body motion, there can be an interchange between the rigid body energy and the elastic energy to the detriment of the flying qualities of the aircraft.

This chapter deals with such effects of structural flexibility, with how they may be described in mathematical terms, and how these terms can be incorporated into the equations of motion of an aircraft. The resulting equations must be used in studies of active control technology and in any studies connected with those special control systems which permit control configured vehicles to produce the performance expected by their designers.

4.2 BENDING MOTION OF THE WING

A wing's lift force, L, is defined by:

$$L = 1/2\,\rho V^2 S C_{L_\alpha} \alpha \tag{4.1}$$

where ρ is the density of the atmosphere, α the angle of attack, S the wing surface area, C_{L_α} the lift curve slope of the wing, and V the speed of the airstream. The dynamic pressure is defined as:

$$\bar{q} = 1/2\,\rho V^2 \tag{4.2}$$

Hence

$$L = \bar{q} S C_{L_\alpha} \alpha = K_w \alpha \tag{4.3}$$

where:

$$K_w = \bar{q} S C_{L_\alpha} \tag{4.4}$$

Equation (4.3) can be represented by Figure 4.1 for all values of α below the stall value, i.e. for all values of angle of attack for which the relationship between lift and angle of attack remains linear.

Figure 4.1 Block diagram representation of an ideal wing.

If a rigid, non-swept, rectangular wing of chord c and semi-span $b/2$ is hinged at its root, as represented in Figure 4.2, the wing has freedom of motion only in bending. The bending angle, λ, is taken as positive when the wing tip is down. The spring has stiffness, K_s, which represents the bending stiffness of the wing in its fundamental mode. The wing also possesses a moment of inertia, I, given by:

$$I = \int_{\text{wing}} \delta m\, y^2 \tag{4.5}$$

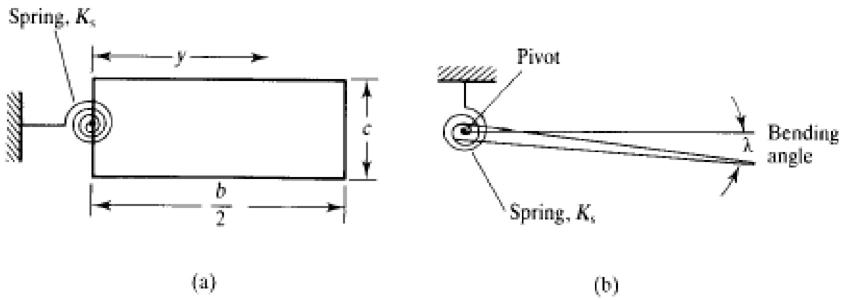


Figure 4.2 Hinged wing.

where δm represents an element of mass. It can be deduced from Figure 4.2(b) that

$$I\ddot{\lambda} + K_s\lambda = 0 \tag{4.6}$$ [1]

where K_s is the bending moment stiffness. This is true only in still air and when structural damping is absent. Equation (4.6) may be re-expressed as:

$$\ddot{\lambda} + \omega^2\lambda = 0 \tag{4.7}$$

where the natural frequency of the bending motion is given by:

$$\omega = (K_s/I)^{1/2} \tag{4.8}$$

When the wing is in a stream of air with relative velocity V, then it can be shown (for example, from quasi-steady aerodynamic strip theory – see Bisplinighoff *et al.*, 1955) that:

$$\begin{aligned} I\ddot{\lambda} + K_s\lambda &= -\, 1/2\,\rho V^2 \int_0^{b/2} \bar{c}y\,\mathrm{d}y\,C_{L_\alpha}\left(\frac{y}{V}\dot{\lambda}\right) \\ &= -\,\bar{q}\,\frac{\bar{c}\,C_{L_\alpha}}{V}\,\dot{\lambda}\int_0^{b/2} y^2\,\mathrm{d}y \\ &= -\,\bar{q}\,\frac{\bar{c}\,C_{L_\alpha}}{V}\,\dot{\lambda}\left[b^3/_{24}\right] \end{aligned} \tag{4.9}$$

Now:

$$S \triangleq b\bar{c}/2 \tag{4.10}$$

$$\therefore I\ddot{\lambda} + K_s\lambda = -\,\frac{K_w b^2}{12V}\,\dot{\lambda} \tag{4.11}$$

i.e.

$$\ddot{\lambda} + \frac{K_w b^2}{12VI}\,\dot{\lambda} + \frac{K_s}{I}\,\lambda = 0 \tag{4.12}$$

or

$$\ddot{\lambda} + 2\,\zeta\omega\dot{\lambda} + \omega^2\lambda = 0 \tag{4.13}$$

where

$$\zeta = \frac{K_w b^2}{24\,V\,\sqrt{(K_s I)}} \tag{4.14}$$

Thus, wing bending motion is characterized by a linear, second order, differential equation in which the damping is provided by aerodynamic forces.

Further discussion of wing flexure can be found in Hancock *et al.* (1985).

4.13 REFERENCES

BISPLINIGHOFF, R.L., H. ASHLEY and R. HALFMAN. 1955. *Aeroelasticity*. Reading, Mass.: Addison Wesley.

HANCOCK, G.J., J.R. WRIGHT and A. SIMPSON. 1985. On the teaching of the principles of wing flexure – torsion flutter. *Aero Journal, RAeS*. Oct., 285–305.

JOHNSON, W. 1981. *Helicopter Theory*. Princeton Univ. Press.

JONES, R.T. 1940. The unsteady lift of a wing of finite aspect ratio. NACA Rpt 681.

McLEAN, D. 1978. Gust alleviation control systems for aircraft. *Proc. IEE*. 125 (7): 675–85.

SCHWANZ, R.C. 1972. Formulations of the equations of motion of an aeroelastic aircraft for stability and control and flight control applications. AFFDL/FGC-TM-72-14. August.

5

Disturbances Affecting Aircraft Motion

5.1 INTRODUCTION

When an aircraft is controlled automatically its motion may be affected by: manoeuvre commands, atmospheric effects, and noise from the system and its sensors.

Manoeuvre commands are applied either by a human pilot or are provided by a guidance, a navigation or a weapons system. Such commands are deliberate inputs to the AFCS, and are intended to change the aircraft's path. The other effects are unwanted disturbances to the aircraft's motion. It is one of the principal functions of an AFCS to suppress as much as possible the unwanted effects of such disturbances. In this chapter only disturbances caused either by atmospheric effects or by sensor noise are considered.

5.2 ATMOSPHERIC DISTURBANCES

The air through which an aircraft flies is never still. As a consequence, whenever an aircraft flies, its motion is erratic. The nature of those disturbances to the air is influenced by many factors, but it is customary to consider turbulence, which occurs above that region where the atmosphere behaves as a boundary layer, as belonging to either of these classes:

1. Convective turbulence, which occurs in and around clouds. This includes thunderstorms particularly.
2. Clear air turbulence (CAT). Below the cloudbase, direct convection heats the air and causes motion which, together with the many small eddies arising as a result of surface heating, are often regarded as mild CAT. Above a cluster of cumulus clouds a regular, short broken motion can persist, particularly when the change in velocity with height is large. More virulent CAT is usually to be found near mountains, and, depending upon the meteorological conditions, flights near the tropopause can often be turbulent. The most virulent turbulence of all, however, is caused by thunderstorms and squall lines, especially when the same area is simultaneously being subjected to rain, hail, sleet or snow.

Another violent atmospheric phenomenon which can be encountered in flight is the microburst, a severe downburst of air. Microbursts are associated with considerable changes in the direction and/or velocity of the wind as the height changes. They exist for only very brief periods. Such severe changes in the nature of the wind over resticted ranges of height are caused by convection and they are often referred to as 'wind shears'. Rising, or falling, columns of air, ringed by toroids of extreme vorticity, are produced by the convection and it is this phenomenon which is called the microburst. A fuller account is presented in Section 5.11.

Because the mechanisms of turbulence are so varied and involved, it has been found that the only effective methods of analysing dynamic problems in which turbulence is involved are statistical methods. However, large gusts, which are reasonably well defined by a particular deterministic function, do occur, but at random times. To assess the effect on the structure of an aircraft encountering such gusts, it is common practice to employ a discrete gust as a load testing function. Even though its time of occurrence may be random, a wind shear can be regarded, once it has occurred, to be effectively a deterministic phenomenon. Thus, in this chapter, there will be presented mathematical models of three types of atmospheric turbulence. The models are not entirely descriptive of the phenomena, but they do represent the significant characteristics sufficiently well to permit an analysis to be carried out with adequate accuracy for engineering purposes. Another method of analysis, which uses an analogue signal in a transient fashion to represent continuous turbulence, is also discussed, before the problem of how the outputs of these models of atmospheric turbulence can be introduced correctly into the equations of motion is dealt with.

The interested reader is referred to Etkin (1980) for further discussion.

5.3 A DISCRETE GUST FUNCTION

That mathematical model, representing a sharp edged gust, which enjoys the most general acceptance for fixed-wing aircraft is the (1-cos) gust, defined thus:

$$x_g(t) = \frac{k}{T}(1\text{-}\cos(2\pi/T)t) \tag{5.1}$$

where the duration of the gust, denoted by T, is given by:

$$T = L/U_0 \tag{5.2}$$

The scale length L is the wavelength of the gust in metres; the equilibrium speed of the aircraft, U_0, is measured in metres per second. In eq. (5.1) k is a scaling factor which is selected to achieve the required gust intensity. The gust function is represented in Figure 5.1. The gust wavelength is traditionally taken to be equal to twenty-five times the mean aerodynamic chord of the wing of the aircraft being considered, i.e.:

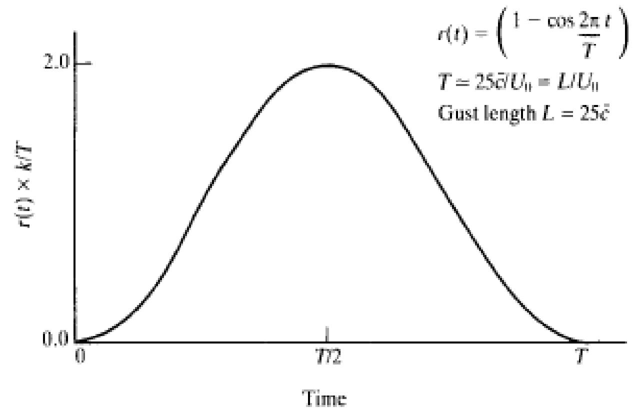


Figure 5.1 (1-cos) gust.

$$L = 25\bar{c} \tag{5.3}$$

This traditional value resulted because study showed that it coupled with the short period pitching and heaving motions of an aircraft to produce the greatest induced load factors. However, as aircraft have flown faster and, as a result of the consequent configuration changes, have become more flexible, it is possible for other gust wavelengths to couple with the flexible modes, thereby producing substantial load responses. When an attempt was made to consider all the possible gust wavelengths which could couple, it became necessary to use statistical methods, particularly the method involving the power spectral density which required a mathematical model to represent the atmospheric turbulence as a stationary, random process. Before dealing with that model, a brief review of the statistical theory associated with the power spectral density functions is presented.

5.8 THE EFFECTS OF GUSTS ON AIRCRAFT MOTION

The components of translational velocity of turbulence are defined as positive along the positive body axes. Hence,

$$\alpha_g = - w_g/U_0 \tag{5.70}$$

$$\beta_g \simeq - v_g/U_0 \tag{5.71}$$

The gust velocities, u_g, w_g and v_g, may vary along the length and span of the aircraft. To account for that it is assumed that the exact distribution of the turbulence velocity over the airframe can be satisfactorily approximated by a truncated Taylor series expansion, i.e.:

$$u_g(x) = u_g(0) + \left.\frac{\partial u_g}{\partial x}\right|_0^x \tag{5.72}$$

$$v_g(y) = v_g(0) + \left.\frac{\partial v_g}{\partial y}\right|_0^y \tag{5.73}$$

$$w_g(x, y) = w_g(0, 0) + \left.\frac{\partial w_g}{\partial x}\right|_0^x + \left.\frac{\partial w_g}{\partial y}\right|_0^y \tag{5.74}$$

For small perturbation motion the gradient of $\partial w_g/\partial x$ is linear and can be taken as the aerodynamic equivalent to the inertial pitching velocity, q. Hence,

$$q_g = \partial w_g/\partial x \tag{5.75}$$

$$p_g = - \partial w_g/\partial y \tag{5.76}$$

$$r_g = \partial v_g / \partial x \tag{5.77}$$

$$\dot{\alpha}_g = \frac{d\alpha_g}{dt} = \frac{\partial \alpha_g}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial}{\partial x}\left(- w_g/U_0\right)U_0 \tag{5.78}$$

$$= - \frac{\partial w_g}{\partial x} \tag{5.79}$$

$$\therefore \quad \dot{\alpha}_g = - q_g \tag{5.80}$$

For example, the equations of small perturbation longitudinal motion, with gust terms included, are given by:

$$\dot{u} = - g\theta + X_\delta \delta + X_u(u + u_g) + X_\alpha(\alpha + \alpha_g) \tag{5.81}$$

$$\begin{aligned} \dot{\alpha} = q &+ \frac{Z_u}{U_0}(u + u_g) + Z_\alpha(\alpha + \alpha_g) + Z_q(q + q_g) \\ &+ Z_{\dot{\alpha}}(\dot{\alpha} - q_g) + Z_\delta \delta \end{aligned} \tag{5.82}$$

$$\begin{aligned} \dot{q} = M_u(u + u_g) &+ M_\alpha(\alpha + \alpha_g) + M_{\dot{\alpha}}(\dot{\alpha} - q_g) \\ &+ M_q(q + q_g) + M_\delta \delta \end{aligned} \tag{5.83}$$

Thus, if:

$$x \triangleq \begin{bmatrix} u \\ \alpha \\ q \\ \theta \end{bmatrix}$$

then

bold

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + EV_g \tag{5.85}$$

where

$$V_g \triangleq \begin{bmatrix} u_g \\ \alpha_g \\ q_g \end{bmatrix}$$

5.12 SENSOR NOISE

Noise on the output signal, which is usually electrical, is regarded as a random signal. However, the properties of these uncertain signals are not well described in the literature and recourse is usually taken in analysis or simulation studies to representing such noise signals as random signals with a Gaussian distribution. They are usually regarded as having been generated as the output signals from linear first order filters which have been driven by white noise sources. The filter time constant is usually selected in the first instance, in the absence of specific knowledge of the power spectral density function relating to the noise, to ensure that the boundaries of the noise spectrum are at least an order greater than those of the AFCS. It is usual to regard the noise signal as being stationary and having zero mean value. But a number of common AFCS sensors are known to have drift rates which, fortunately, are very slow. For example, a typical attitude gyro may have a drift rate of $0.1° \text{ h}^{-1}$, (4.84×10^{-7} rad s^{-1}). The accuracy of gyroscopes is typically 0.1°, or $0.1° \text{ s}^{-1}$, if it is a rate gyroscope. Accelerometers have errors of, typically, 3×10^{-5} g (3×10^{-3} m s^{-2}), and barometric altimeters are subject to typical r.m.s. errors of 16 m. For accelerometers, a typical r.m.s. noise figure is 10^{-4} g (10^{-3} m s^{-2}) with the corresponding power spectral density being approximately 3×10^{-7} g^2/Hz.

5.13 CONCLUSIONS

This chapter presents some information about the disturbances which most affect the operation of AFCSs. The atmospheric turbulence phenomena considered were continuous gusts, defined by the mathematical model suggested by Dryden as a practical improvement on the Von Karman model, and the discrete (1 − cos) gust. Generating test signals by means of a transient analogue was also dealt with. An account of wind shear and some methods of representing such a phenomenon, particularly the microburst, was also given and the chapter closed with a brief note on the nature of representations of sensor noise.

6

Flying and Handling Qualities

6.1 INTRODUCTION

A special issue of the influential *Journal of Guidance, Control and Dynamics* from the American Institute of Aeronautics and Astronautics was concerned with aircraft flying qualities which were defined in the editorial as 'those qualities of an aircraft which govern the ease and precision with which a pilot is able to perform his mission'. All the papers which made up that special issue refer to the handling qualities of the aircraft. It is helpful to those new to the field to distinguish between flying and handling qualities; with experience, the two will be seen to merge into a single topic.

Aircraft flying qualities are usually characterized by a number of parameters relating to the complex frequency domain, such as the damping ratio and undamped natural frequency of the short period longitudinal motion of the aircraft. Knowledge of these parameters allows a designer to imagine the nature of the aircraft's response to any command or disturbance; it allows a general notion of how the aircraft will fly in a controlled manner.

Handling qualities reflect the ease with which a pilot can carry out some particular mission with an aircraft which has a particular set of flying qualities. However, handling qualities depend not only upon flying qualities but also upon the primary flying controls, the visual and motion cues available, and the display of flight information in the cockpit. The importance of handling qualities is particularly marked when some aircraft exhibit such unwanted flight characteristics as pilot-induced oscillations or roll ratchet. It should always be remembered that a human pilot is a variable, dynamic element closing an outer loop around an AFCS. Handling qualities ought to be arranged, therefore, to suit the pilot, so that his adapted characteristic is best for the flight mission. Sometimes, special command input filters are added to AFCSs to assist in providing acceptable handling qualities. Since different types of aircraft can carry out similar missions, it follows that the required handling qualities also depend upon the type of aircraft.

Extensive research into flying and handling qualities has been carried out in many countries for a great number of years. Harper and Cooper (1986) provide an excellent account of this research. The results of these studies have been incorporated into specifications for aircraft flying qualities which have been laid down by the statutory bodies responsible for aviation in different countries. Although ten years ago, the UK specifications were in a number of respects different in expression from those laid down by the American authorities, it was decided by 1978 that the UK specifications (MoD, 1983) should correspond wherever possible with those used by the American authorities. For most classes of fixed wing aircraft, the most significant of these specifications is MIL-F-8785(ASG), Military Specification – Flying Qualities of Piloted Airplanes published in 1980. If general aviation aircraft are to be considered, the specification generally used is FAR 23 issued by the Federal Aviation Authority (FAA) in the USA. Whenever AFCS designs are to be studied, then it is necessary to consider, in conjunction with MIL-F-8785(ASG), other specifications laid down by the American military authorities, namely MIL-F-9490D (see references at end of chapter), which is the current USAF flight controls specification, and MIL-C-18244, which is a general specification for piloted airplanes with automatic control and stabilization systems. The appropriate specification defining the flying and ground handling qualities for military helicopters is MIL-H-8501A. When the concern is VSTOL aircraft then the appropriate specification is MIL-F-83300. Details can be found in the references at the end of this chapter.

In this book, it is essentially the recommendations of MIL-F-8785 which are followed for fixed wing aircraft, and those of MIL-H-8501A for rotary wing aircraft. Since many of the specifications in MIL-F-8785 are framed with reference to aircraft classes, flight phases, and levels of flying qualities, these terms are explained first before discussing the specifications.

6.2 SOME DEFINITIONS REQUIRED FOR USE WITH FLYING QUALITIES' SPECIFICATION

6.2.1 Aircraft Classes

An aircraft is considered to belong to one of the four classes shown in Table 6.1.

Table 6.1 Aircraft classification

Class	*Aircraft characteristics*
I	Small, light aircraft (max. weight = 5 000 kg)
II	Aircraft of medium weight and moderate manoeuvrability (weight between 5 000 and 30 000 kg)
III	Large, heavy aircraft with moderate manoeuvrability (30 000+ kg)
IV	Aircraft with high manoeuvrability

6.2.2 Flight Phases

Whatever mission an aircraft is used to accomplish, the mission is divisible into three phases of flight, as follows:

Phase A which includes all the non-terminal phases of flight such as those involving rapid manoeuvring, precision tracking, or precise control of the flight path. Included in phase A would be such flight phases as: air-to-air combat (CO), ground attack (GA), weapon delivery (WD), reconnaissance (RC), air-to-air refuelling in which the aircraft acts as the receiver (RR), terrain following (TF), maritime search and rescue (MS), close formation flying (FF), and aerobatics (AB).

Phase B involves the non-terminal phases of flight usually accomplished by gradual manoeuvres which do not require precise tracking. Accurate flight path control may be needed, however. Included in the phase would be: climbing (CL), cruising (CR), loitering (LO), descending (D), aerial delivery (AD) and air-to-air refuelling in which the aircraft acts as a tanker (RT).

Phase C involves terminal flight phases, usually accomplished by gradual manoeuvres, but requiring accurate flight path control. This phase would include: take-off (TO), landing (L), overshoot (OS) and powered approach (including instrument approach) (PA).

6.2.3 Levels of Acceptability

The requirements for airworthiness are stated in terms of three distinct, specified values of control (or stability) parameter. Each value is a limiting condition necessary to satisfy one of the three levels of acceptability. These levels are related to the ability to complete the missions for which the aircraft is intended. The levels are defined in Table 6.2.

Table 6.2 Flying level specification

Level	*Definition*
1	The flying qualities are completely adequate for the particular flight phase being considered.
2	The flying qualities are adequate for the particular phase being considered, but there is either some loss in the effectiveness of the mission, or there is a corresponding increase in the workload imposed upon the pilot to achieve the mission, or both.
3	The flying qualities are such that the aircraft can be controlled, but either the effectiveness of the mission is gravely impaired, or the total workload imposed upon the pilot to accomplish the mission is so great that it approaches the limit of his capacity.

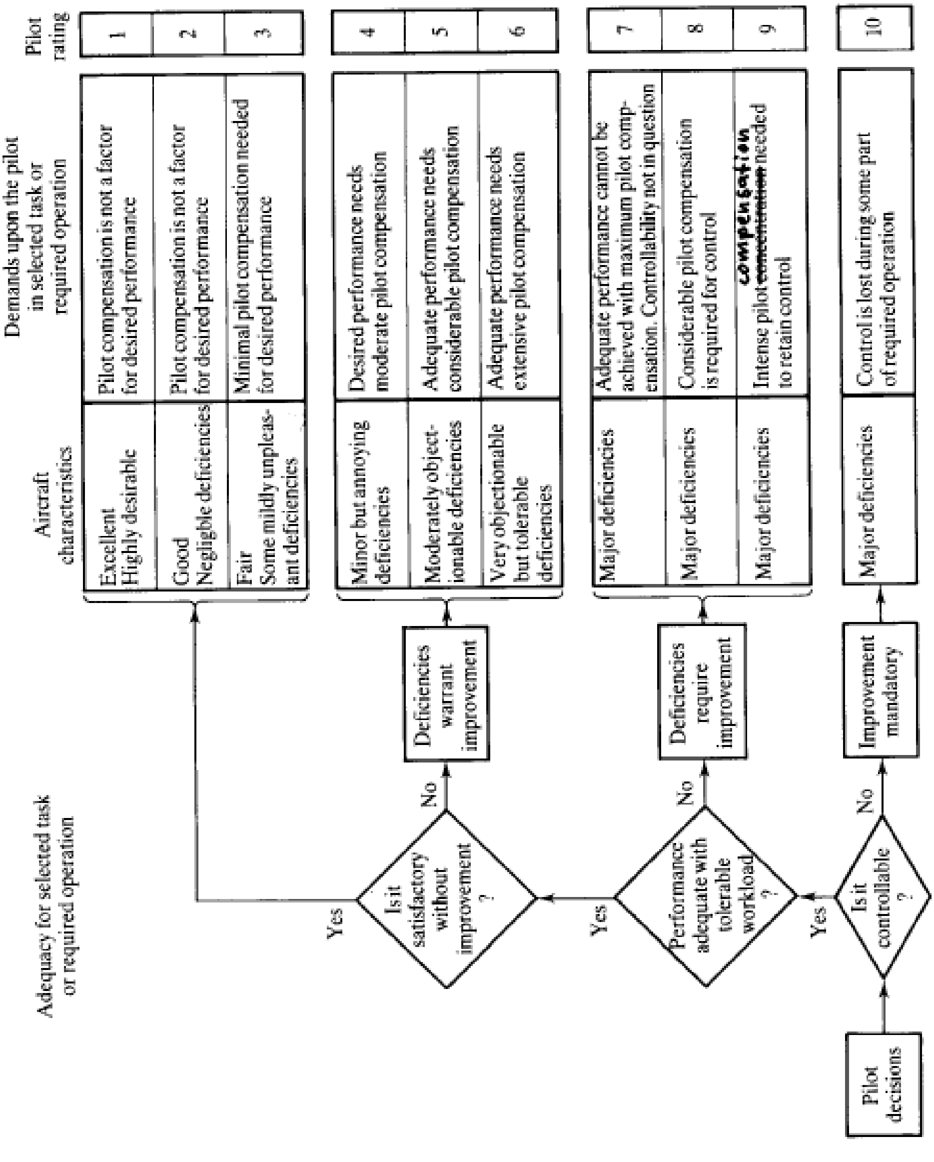


Figure 6.1 Cooper–Harper rating chart.

There is a direct relationship between these levels of acceptability and the pilot rating scale developed by Cooper and Harper (1986). The rating scale is shown in Figure 6.1 and a representation of the relationship between the rating scale and the levels of acceptability is illustrated in Figure 6.2

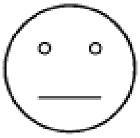
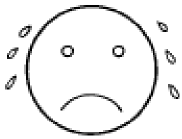
Pilot state	Pilot rating	Level	Definition
	1 3½	1	Clearly adequate for the mission flight phase
	6½	2	• Adequate to accomplish mission flight phase • Increase in pilot workload, or loss of effectiveness of mission, or both
	9 10	3	• Aircraft can be controlled • Pilot workload excessive – mission effectiveness impaired • Category A flight phases can be terminated safely

Figure 6.2 Acceptable level of flying qualities.

6.3 LONGITUDINAL FLYING QUALITIES

6.3.1 Static Stability

An aircraft should have no tendency for its airspeed to diverge aperiodically whenever it is disturbed from its trim condition and with its pitch control either free or fixed.

6.3.2 Phugoid Response

Provided that the frequencies of the phugoid and the short period modes of motion are widely separated, for the pitch control either being free or fixed, the values of damping ratio quoted in Table 6.3 must be achieved.

If the separation between the frequencies of the phugoid and short period modes is small, handling difficulties can arise. If $\omega_{ph}/\omega_{sp} < 0.1$ there may be some trouble with the handling qualities.

Table 6.3 Phugoid mode flying qualities

Level	*Damping ratio of phugoid mode*
1	$\geqq 0.04$
2	$\geqq 0.0$
3	An undamped oscillatory mode ~~having a period~~ of at least 55 s with a time to double amplitude

6.3.3 Short Period Response

The flying qualities related to this work are governed by the parameters, ζ_{sp}, the short period damping ratio, and ω_{sp}/n_{z_α} where n_{z_α} is the acceleration sensitivity of the aircraft. The specified values of damping ratio are quoted in Table 6.4. At high speed, low values of short period damping ratio are less troublesome than at low speeds.

Table 6.4 Short period mode damping ratio specification

Flight phase category	*Level 1*		*Level 2*		*Level 3*	
	Min.	*Max.*	*Min.*	*Max.*	*Min.*	*Max.*
A	0.35	1.3	0.25	2.0	0.15	—
B	0.3	2.0	0.2	2.0	0.15	—
C same as A	~~0.5~~	~~—~~	~~0.35~~	~~2.0~~	~~0.25~~	~~—~~

If the short period oscillations are non-linear with amplitude, then the flying qualities parameters quoted must apply to each cycle of the oscillation.

The specified limits for the undamped natural frequency are functions of the acceleration sensitivity, n_{z_α}, for any particular level category and phase; the specification is usually presented as a figure such as Figure 6.3.

The curves defining the upper and lower frequency limits are straight lines, each with a slope of + 0.5 on the log-log plot. The parameter $\omega_{sp}^2/n_{z_\alpha}$ is referred to as the control anticipation parameter (CAP) which relates initial pitch acceleration to steady state normal load factor, i.e.:

$$\text{CAP} = \dot{q}(0)/n_{z_{cg|ss}} \tag{6.1}$$

This parameter has been proposed upon the assumption that when a pilot initiates a manoeuvre the response of greatest importance to him is the initial pitch acceleration. In a pull-up manoeuvre, his concern is with the steady state normal acceleration. By assuming constant speed flight, and by applying to the approximate transfer function relating pitch acceleration to an elevator deflection

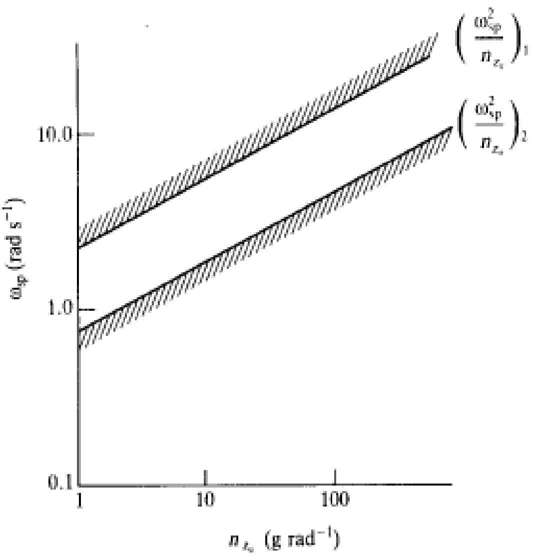


Figure 6.3 Handling qualities diagram.

the initial value theorem, and then the final value theorem to the transfer function relating normal acceleration, $n_{z_{cg}}$, to the same elevator input, an expression for the CAP can be written, if it is assumed that the elevator deflection is a step input:

$$\begin{aligned} n_{z_{cg}}\Big|_{ss} &= U_0(Z_{\delta_E}M_w - M_{\delta_E}Z_w)/g\omega_{sp}^2 \\ &\simeq M_{\delta_E}n_{z_\alpha}/\omega_{sp}^2 \end{aligned} \tag{6.2}$$

If:

$$\frac{sq(s)}{\delta_E(s)} \triangleq \frac{M_{\delta_E}}{(s + (1/T_E)} \tag{6.3}$$

where $T_E^{-1} \simeq -M_q$ then:

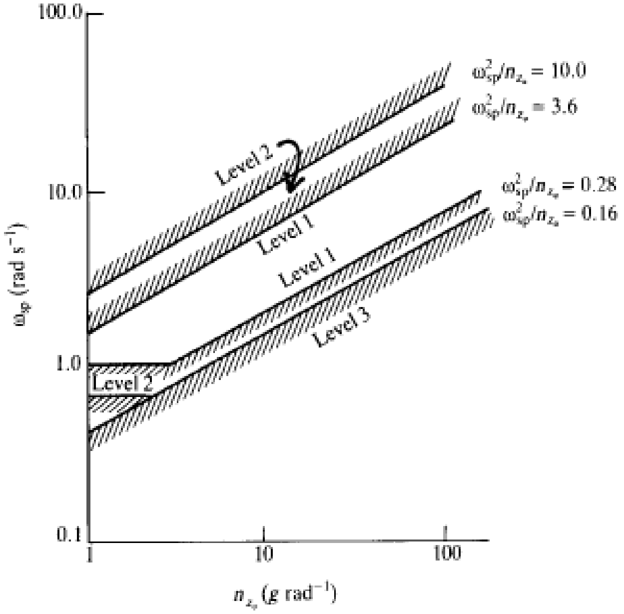
$$\dot{q}(0) \to M_{\delta_E} \tag{6.4}$$

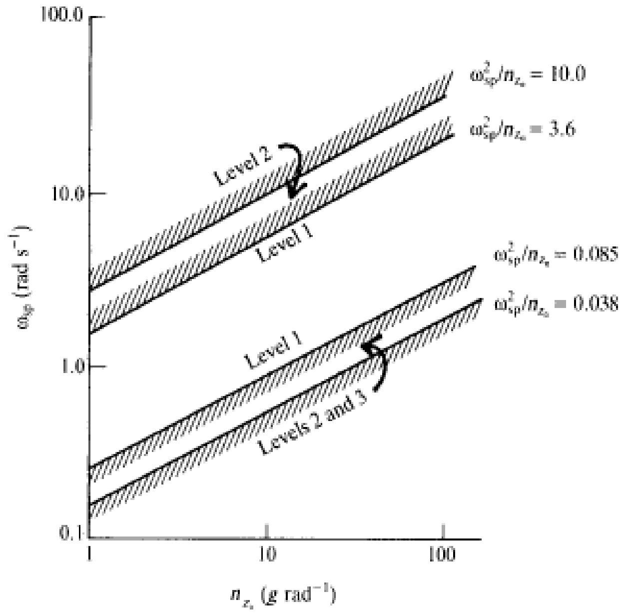
and the CAP defined in eq. (6.1) is obtained.

Figure 6.4 shows the specifications for levels 1, 2 and 3 for categories A, B, and C.

6.4 LATERAL/DIRECTIONAL FLYING QUALITIES

The specification of flying qualities for lateral/directional motion is more involved than for longitudinal motion and, consequently, requires more parameters.


100.0
10.0
1.0
0.1
$\omega_{sp}^2/n_{z_\alpha} = 10.0$
$\omega_{sp}^2/n_{z_\alpha} = 3.6$
$\omega_{sp}^2/n_{z_\alpha} = 0.28$
$\omega_{sp}^2/n_{z_\alpha} = 0.16$
Level 2
Level 1
Level 1
Level 3
Level 2
ω_{sp} (rad s^{-1})
1
10
100
n_{z_α} (g rad^{-1})


100.0
10.0
1.0
0.1
$\omega_{sp}^2/n_{z_\alpha} = 10.0$
$\omega_{sp}^2/n_{z_\alpha} = 3.6$
$\omega_{sp}^2/n_{z_\alpha} = 0.085$
$\omega_{sp}^2/n_{z_\alpha} = 0.038$
Level 2
Level 1
Level 1
Levels 2 and 3
ω_{sp} (rad s^{-1})
1
10
100
n_{z_α} (g rad^{-1})

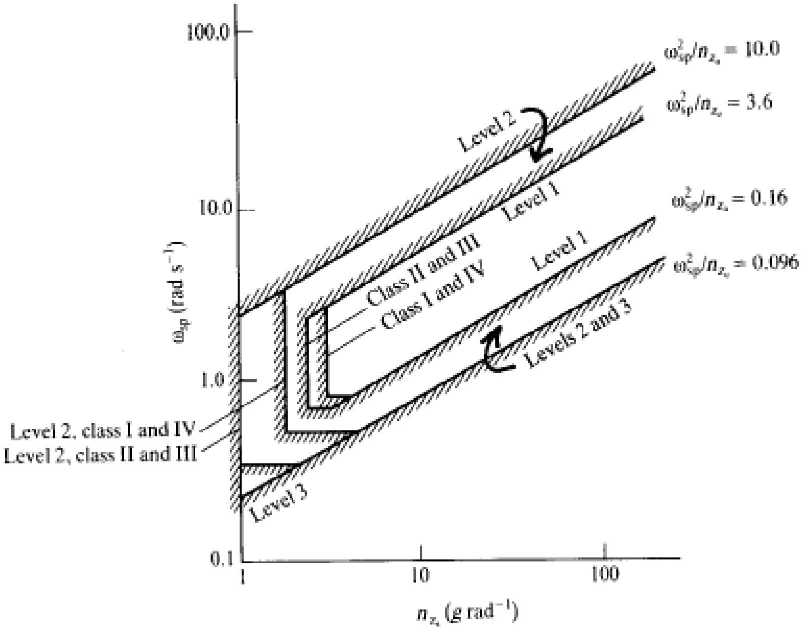


Figure 6.4 Short period frequency requirements. (a) CAT. A. (b) CAT. B. (c) CAT. C.

6.4.1 Rolling Motion

The time constant of the roll subsidence mode, T_R, is required to be less than the specified maximum values given in Table 6.5. It is customary to specify roll performance in terms of the change of bank angle achieved in a given time in response to a step function in roll command. The required bank angles and time are specified in Table 6.6.

Table 6.5 Roll mode time constant specification

Flight phase category	*Class*	*T_R (seconds)*			
		Level 1	*Level 2*	*Level 3*	
A	I, IV	1.0	1.4	~~Not specified~~ –	10.
A	II, III	1.4	3.0	~~limit is believed~~	10.
B	All	1.4	3.0	~~to lie within~~	10.
C	I, IV	1.0	1.4	~~range 6–8 s~~	10.
C	II, III	1.4	3.0		10.

Table 6.6 Bank angle specification

Class	*Flight phase category*	*Bank angle in fixed time*		
		Level 1	*Level 2*	*Level 3*
	A	60° in 1.3 s	60° in 1.7 s	60° in 2.6 s
I	B	60° in 1.7 s	60° in 2.5 s	60° in 3.4 s
	C	30° in 1.3 s	30° in 1.8 s	30° in 2.6 s
	A	45° in 1.4 s	45° in 1.9 s	45° in 2.8 s
II	B	45° in 1.9 s	45° in 2.8 s	45° in ~~[illegible]~~ s 3.8
	C	30° in ~~[illegible]~~ s 1.8	30° in ~~[illegible]~~ s 2.5	30° in ~~[illegible]~~ s 3.6
	A	30° in 1.5 s	30° in 2.0 s	30° in 3.0 s
III	B	30° in 2.0 s	30° in ~~[illegible]~~ s 3.3	30° in ~~[illegible]~~ s 5.0
	C	30° in ~~[illegible]~~ s 2.5	30° in 4.0 s	30° in 6.0 s
	A	90° in 1.3 s	90° in 1.7 s	90° in 2.6 s
IV	B	90° ~~60°~~ in 1.7 s	90° ~~60°~~ in 2.5 s	90° ~~60°~~ in 3.4 s
	C	30° in ~~[illegible]~~ s 1.1	30° in 1.3 s	30° in 2.0 s

For class IV aircraft, for level 1, the yaw control should be free. For other aircraft and levels it is permissible to use the yaw control to reduce any sideslip which tends to retard roll rate. Such yaw control is not permitted to induce sideslip which enhances the roll rate.

6.4.2 Spiral Stability

When specifying spiral stability it is assumed that the aircraft is trimmed for straight and level flight, with no bank angle, no yaw rate and with the flying controls free. The specification is given in terms of the time taken for the bank angle to double following an initial disturbance in bank angle of up to 20°. The time taken must *exceed* the values given in Table 6.7.

Table 6.7 Spiral mode stability specification

Flight phase category	*Level*		
	1	2	*3*
A and C	12 s	8 s	~~5 s~~ 4 s
B	20 s	8 s	~~5 s~~ 4 s

Table 6.8 Dutch roll mode specification

Flight phase category zeta not xi	*Class*	*Level*								
		1			*2*			*3*		
ζ_D →		ξ_D	$\xi_D\omega_D$ rad/s	ω_D rad/s	ξ_D	$\xi_D\omega_D$ rad/s	ω_D rad/s	ξ_D	$\xi_D\omega_D$ rad/s	ω_D rad/s
A	I, IV	0.19	0.35	1.0	0.02	0.05	0.4	0.02	—	0.4
A	II, III	0.19	0.35	0.4	0.02	0.05	0.4	0.02	—	0.4
B	All	0.08	0.15	0.4	0.02	0.05	0.4	0.02	—	0.4
C	I, IV	0.08	0.15	1.0	0.02	0.05	0.4	0.02	—	0.4
C	II, III	0.08	0.1	0.4	0.02	0.05	0.4	0.02	—	0.4

Note: minimum values are specified
the governing damping requirement equals the Largest value of ζ_D obtained from either of the two columns Labled ζ_D and $\zeta_D \cdot \omega_D$.

6.4.3 Lateral/Directional Oscillations – Dutch Roll

Although the dutch roll mode has very little useful part to play in the control of an aircraft, it does have significant nuisance value. The values of the important dutch roll parameters, namely damping ratio, ζ_D, the dutch roll frequency, ω_D, are specified in Table 6.8.

It is usual to avoid coupled roll/spiral oscillation as its leads to inferior tracking performance.

For atmospheric turbulence the Tables 6.5, 6.7 and 6.8 are still valid. For bank angle, however, for a class IV aircraft, level 1, category A flight phase, the r.m.s. value of bank angle which arises in severe turbulence must be less than 2.7°.

6.5 THE C^* CRITERION

This criterion can be used to assess the dynamic response of the aircraft's longitudinal motion to a manoeuvre command. When an AFCS is used, it has been found that if the poles and zeros of the controller are located in the s-plane such that they are close in frequency to the resulting short period frequency, ω_{sp}, of the uncontrolled aircraft, the resulting dynamic response of the controlled aircraft is so altered that characterizing the response by specifying the short period damping ratio and undamped natural frequency is unsatisfactory. The C^* criterion is based upon the tailoring of the total response of the controlled aircraft to pilot inputs such that the defined output response lies between specific limits. The quantity C^* is a measure of a blended contribution to the total response from the normal acceleration, the pitch acceleration, and the pitch rate of the aircraft. That blend varies with airspeed; the acceleration measure, $C^*(t)$, is arranged so

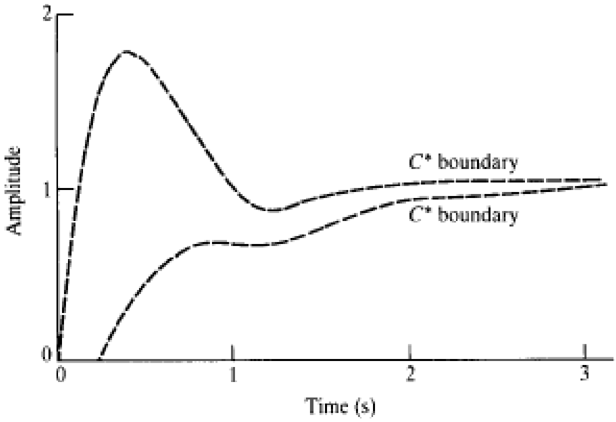


Figure 6.5 C^* time history for category A.

that when the crossover speed, U_c, is reached, the contributions to $C^*(t)$ from the normal acceleration term and the terms related to pitching motion are equal. The crossover speed is a weighting factor which reflects the change in emphasis which pilots place on motion cues at certain speeds, a change from controlling pitch rate at the lower speeds to an emphasis upon controlling normal acceleration at the higher speeds. One definition of $C^*(t)$ is:

$$C^*(t) = n_z l_{x_{pilot}} + (U_c/g) \quad n_z + \frac{U_c}{g}\cdot q \tag{6.5}$$

The criterion adopted is that the normalized time response, $C^*(t)/C^*_{ss}$, shall lie between two specified boundaries. For as long as the $C^*(t)/C^*_{ss}$ response remains within the specified boundaries the AFCS designer may assume that the response of the controlled aircraft is satisfactory, without regard to the details of the control system or the aircraft dynamics being considered. Typical C^* boundaries are shown in Figure 6.5, for flight condition 1. Similar boundaries obtain for the other flight categories.

It must be remembered that C^* is a function of time and, consequently, the C^* criterion is a performance criterion for the time domain. It should be noted that C^* can be treated as an output variable of the aircraft. In Chapter ~~3~~ 2 it is shown that n_{z_x} could be expressed as:

$$n_{z_x} = \frac{1}{g}\left\{a_{z_{cg}} - l_x\dot{q}\right\} \tag{6.6}$$

$$\therefore \; n_{z_{pilot}} = \frac{1}{g}\left\{Z_u u + Z_w w + Z_{\delta_E}\delta_E - l_{x_{pilot}}(\tilde{M}_u u + \tilde{M}_w w + \tilde{M}_q q + \tilde{M}_{\delta_E}\delta_E)\right\}$$

$$= \frac{1}{g}\left\{[Z_u - l_{x_{pilot}}\tilde{M}_u]u + [Z_w - l_{x_{pilot}}\tilde{M}_w]w - l_{x_{pilot}}\tilde{M}_q q + [Z_{\delta_E} - l_{x_{pilot}}\tilde{M}_{\delta_E}]\delta_E\right\} \tag{6.7}$$

Hence,

$$y \triangleq n_{z_{x_{\text{pilot}}}} = C\mathbf{x} + D\mathbf{u} \qquad (6.8)$$

where:

$$\mathbf{x}' \triangleq [u \ w \ q \ \theta]$$

$$\mathbf{u} \triangleq \delta_E$$

$$C = \left[\frac{(Z_u - l_{x_{\text{pilot}}} \tilde{M}_u)}{g} \ \frac{(Z_w - l_{x_{\text{pilot}}} \tilde{M}_w)}{g} \ - \frac{l_{x_{\text{pilot}}} \tilde{M}_q}{g} \ 0 \right]$$

$$D = \left[\frac{Z_{\delta_E}}{g} \ \frac{l_{x_{\text{pilot}}} \tilde{M}_{\delta_E}}{g} \right]$$

There is still uncertainty about the general applicability of the C^* criterion, however. The problem can be seen from Figure 6.6 from which it is seen that system 1 has a number of overshoots, but lies wholly within the boundaries. System 2 infringes the boundary slightly at the initial part of the response. System 1 attracted a pilot rating of 8.5 and system 2 was awarded 2.5.

It is this difficulty of reconciling human prejudices with quantitative performance indices and parameters which makes the study of handling and flying qualities a most demanding and protracted technical problem. The single fact which ~~it~~ is essential for students to understand is that extensive studies related to the flying qualities specification must be undertaken, before being satisfied that any AFCS design is acceptable; it must never be forgotten that the motion of an aircraft is controlled by a number of control surfaces which a pilot, human or automatic, can operate simultaneously.

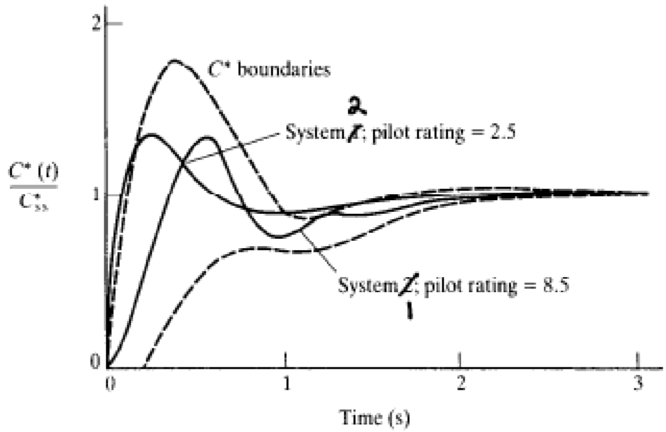


Figure 6.6 Different C^* responses with pilot rating.

6.8 CONCLUSIONS

This chapter introduces the important subjects of the flying and handling qualities of an aircraft. They are important because they involve a set of complex interactions between the pilot, the aircraft, the operational environment and the mission which is being flown. Since these qualities are what govern the ease, the accuracy, and the precision with which a pilot can carry out his flying task it is specially important for the designer of an AFCS to understand them, how they are specified and how they can be measured, for, if an aircraft has been found to have poor handling qualities, it is customary to recover the loss by introducing a control system. The importance of these aircraft qualities is not lessened by the introduction of modern technology; indeed, with the introduction of digital flight control systems the inevitable time delays involved in this form of control law generation invariably have a detrimental effect on aircraft handling.

The reader should regard this chapter as no more than a brief introduction to a complex scientific study which is more fully accounted for in the papers by Harper and Cooper (1986) and McRuer *et al.* (1962), and the special issue of the *Journal of Guidance, Control and Dynamics* (1986).

6.10 REFERENCES

Department of Defense. Flight control systems – general specification for design, installation and test of piloted aircraft. MIL-F-9490D. Washington, D.C.: DoD.

Department of Defense. 1961. Helicopter flying and ground qualities: general requirements. MIL-H-8501A. Washington, D.C.: DoD.

Department of Defense. 1970. Flying qualities of piloted VSTOL aircraft. MIL-F-83300. Washington, D.C.: DoD.

Department of Defense. 1980. Military specifications – flying qualities of piloted airplanes. MIL-F-8785C. Washington, D.C.: DoD.

HARPER, R.P. and G.E. COOPER. 1986. Handling qualities and pilot evaluation – 1984 Wright Bros. Lecture. *J. Guid. Cont.* 9(5): 515–29.

McRUER, D.T. and I.L. ASHKENAS. 1962. Design implications of the human transfer function. *Aerospace Eng.* 21: 76–7, 144–7.

Ministry of Defence, 1983. Design and airworthiness requirements for service aircraft. Def. Standard 00–970, Vol. 1, Aeroplanes. London: MoD.

J. Guid., Cont. and Dyn. 1986. 9(5). Special Issue on Aircraft Flying Qualities.

Appendix B

Stability Derivatives for Several Representative Modern Aircraft

B.1 NOMENCLATURE

Some stability data for seven aircraft are presented here. These aircraft are generic types and are referred to as follows:

ALPHA	a four-engined, executive jet aircraft
BRAVO	a twin-engined, jet fighter aircraft
CHARLIE	a very large, four-engined, passenger jet aircraft
DELTA	a very large, four-engined, cargo jet aircraft
ECHO	a single-engined, CCV, jet fighter aircraft
FOXTROT	a twin-engined, jet fighter/bomber aircraft
GOLF	a twin-piston engined, general aviation aircraft

When referring to an aircraft and its particular flight condition, the aircraft name is given first followed by a number corresponding to the flight condition. For example, FOXTROT-3 means flight condition 3 for the aircraft, FOXTROT.

B.2 AIRCRAFT DATA

B.2.1 ALPHA – A four-engined, executive jet aircraft

General Parameters

Wing area (m^2)	50.4
Aspect ratio:	5.325
Chord, $\bar{c}$ (m):	3.33
Total related thrust (kN):	59.2
C.g.:	$0.25\bar{c}$
Pilot's location (m) (relative to c.g.)	
l_{x_p}:	6.77
l_{z_p}:	– 0.73

Weight (kg):	*Approach* 10 635	*All other flight conditions* 17 000
Inertias ($kg\,m^2$)		
I_{xx}:	57 000	162 000
I_{yy}:	171 500	185 000
I_{zz}:	218 500	330 000
I_{xz}:	7 500	6 900

Flight Conditions

Parameter	*Flight condition*			
	1	*2*	*3*	*4*
Height (m)	S.L.	6 100	6 100	12 200
Mach no.	0.2	0.35	0.75	0.8
U_0 ($m\,s^{-1}$)	67.7	110.6	237.1	236.0
$\bar{q}$ ($N\,m^{-2}$)	2 844.0	4 000	18 338	8 475
α_0 (degrees)	+ 6.5	+ 9.9	+ 2.6	+ 4.2
γ_0 (degrees)	0	0	0	0

Stability Derivatives

Longitudinal Motion

Stability derivative	*Flight condition*			
	1	*2*	*3*	*4*
X_u	-0.0166	-0.00324	-0.0157	-0.211×10^{-5}
X_w	0.108	0.00102	-0.0005	-0.0043
X_{δ_E}	0.6	0.8	1.02	0.774
X_{δ_T}	0.92×10^{-4}	5.73×10^{-5}	5.73×10^{-5}	5.73×10^{-5}
Z_u	-0.175	-0.08	-0.02	-0.035
Z_w	-1.01	-0.565	-1.33	-0.665
Z_{δ_E}	-5.24	-4.57	-22.4	-10.55
M_u	0.0043	0.0033	-0.0015	-0.014
M_w	-0.033	-0.022	-0.051	-0.025
$M_{\dot{w}}$	-0.003	-0.0015	-0.002	-0.001
M_q	-0.546	-0.439	-1.09	-0.506
M_{δ_E}	-2.26	-2.95	-14.5	-6.78
M_{δ_T}	-0.65×10^{-5}	-0.6×10^{-5}	-0.6×10^{-5}	-0.6×10^{-5}

Lateral Motion

Stability derivative	Flight condition 1	2	3	4
Y_v	− 0.014	− 0.076	− 0.167	− 0.078
$Y^*_{\delta_R}$	0.034	0.018	0.037	0.016
L'_β	− 4.05	− 3.23	− 4.93	− 2.27
L'_p	− 1.85	− 0.58	− 1.34	− 0.64
L'_r	0.52	0.17	0.09	0.06
L'_{δ_A}	2.21	1.1	5.83	2.64
L'_{δ_R}	1.11	0.57	2.43	1.21
N'_β	1.34	1.21	5.63	2.66
N'_p	− 0.25	− 0.12	− 0.14	− 0.07
N'_r	− 0.19	− 0.125	− 0.25	− 0.12
N'_{δ_A}	− 0.006	− 0.08	− 0.06	− 0.072
N'_{δ_R}	− 0.64	− 0.62	− 2.66	− 1.16

B.2.2 BRAVO – A twin-engined, jet fighter aircraft

General Parameters

Wing area (m^2):	56.5	
Aspect ratio:	3.0	
Chord, $\bar{c}$ (m):	4.86	
Total related thrust (kN):	210 (no reheat)	
C.g.:	0.255 $\bar{c}$ or 0.311 $\bar{c}$	
Pilot's location (m) (relative to c.g.)		
l_{x_p}:	8.2	
l_{z_p}:	− 1.3	
Weight (kg):	*Approach*	*All other flight conditions*
	15×10^3	16×10^3
Inertias (kg m^2):		
I_{xx}:	35 250	38 000
I_{yy}:	176 250	255 000
I_{zz}:	210 000	285 000
I_{xz}:	3 000	4 000

Flight Conditions

Parameter	*Flight condition*			
	1	*2*	*3*	*4*
Height (m)	S.L.	6 100	6 100	9 150
Mach no.	0.4	0.6	0.6	0.8
U_0 (m s^{-1})	136	190	190	240
$\bar{q}$ (N m^{-2})	11 348	11 760	11 760	10 700
α_0 (degrees)	+ 3.5	+ 8.5	+ 8.5	+ 2.5
γ_0 (degrees)	0	0	0	0
c.g.	0.311	0.255	0.311	0.311

Stability Derivatives

Longitudinal Motion only

Stability derivative	*Flight condition*			
	1	*2*	*3*	*4*
X_u	− 0.017	− 0.011	− 0.012	− 0.007
X_α	0.026	0.018	0.017	0.012
Z_u	− 0.143	− 0.113	− 0.113	− 0.128
Z_α	− 1.02	− 0.72	− 0.72	− 0.54
Z_q	− 0.0076	− 0.0044	− 0.0044	− 0.0027
Z_{δ_E}	− 0.064	− 0.047	− 0.047	− 0.036
M_u	0	0	0	0
M_α	1.4	− 2.7	1.09	0.69
$M_{\dot{\alpha}}$	− 0.66	− 0.61	− 0.54	− 0.51
M_q	− 0.53	− 0.64	− 0.57	− 0.48
M_{δ_E}	− 11.56	− 13.04	− 12.25	− 12.63

B.2.3 CHARLIE – A very large, four-engined, passenger jet aircraft

General Parameters

Wing area (m^2):	510
Aspect ratio:	7.0
Chord, $\bar{c}$ (m):	8.3
Total related thrust (kN):	900
C.g.:	0.25 $\bar{c}$

Pilot's location (m)
(relative to c.g.)

l_{x_p}:	26.2	
l_{z_p}:	− 3.05	
Weight (kg):	*Approach*	*All other flight conditions*
	250 000	290 000
Inertias (kg m^2):		
I_{xx}:	18.6×10^6	24.6×10^6
I_{yy}:	41.35×10^6	45×10^6
I_{zz}:	58×10^6	67.5×10^6
I_{xz}:	1.2×10^6	1.32×10^6

Flight Conditions

Parameter	*Flight condition*			
	1	*2*	*3*	*4*
Height (m)	S.L.	6 100	6 100	12 200
Mach no.	0.198	0.5	0.8	0.8
U_0 (m s^{-1})	67	158	250	250
$\bar{q}$ (N m^{-2})	2 810	8 667	24 420	9 911
α_0 (degrees)	8.5	6.8	0	4.6
γ_0 (degrees)	0	0	0	0

Stability Derivatives

Longitudinal Motion

Stability derivative	*Flight condition*			
	1	*2*	*3*	*4*
X_u	− 0.021	0.003	− 0.0002	0.0002
X_w	0.122	0.078	0.026	0.039
X_{δ_E}	0.292	0.616	0.0	0.44
$X_{\delta_{th}}$	3.88×10^{-6}	3.434×10^{-6}	3.434×10^{-6}	3.434×10^{-6}
Z_u	− 0.2	− 0.07	− 0.09	− 0.07
Z_w	− 0.512	− 0.433	− 0.624	− 0.317
Z_q	− 1.9	− 1.95	− 3.04	− 1.57
Z_{δ_E}	− 1.96	− 5.15	− 8.05	− 5.46
$Z_{\delta_{th}}$	-1.69×10^{-7}	-1.5×10^{-7}	-1.5×10^{-7}	-1.5×10^{-7}
M_u	0.000036	0.00008	− 0.00007	0.00006
M_w	− 0.006	− 0.006	− 0.005	− 0.003
$M_{\dot{w}}$	− 0.0008	− 0.0004	− 0.0007	− 0.0004

Longitudinal Motion Cont'd

Stability derivative	*Flight condition* 1	2	3	4
M_q	− 0.357	− 0.421	− 0.668	− 0.339
M_{δ_E}	− 0.378	− 1.09	− 2.08	− 1.16
$M_{\delta_{th}}$	0.7×10^{-7}	0.67×10^{-7}	0.67×10^{-7}	0.67×10^{-7}

Lateral Motion

Stability derivative	*Flight condition* 1	2	3	4
Y_v	− 0.089	− 0.082	− 0.12	− 0.056
$Y^*_{\delta_R}$	0.015	0.014	0.014	0.012
L'_β	− 1.33	− 2.05	− 4.12	− 1.05
L'_p	− 0.98	− 0.65	− 0.98	− 0.47
L'_r	+ 0.33	+ 0.38	+ 0.29	+ 0.39
L'_{δ_A}	0.23	0.13	0.31	0.14
L'_{δ_R}	0.06	0.15	0.18	0.15
N'_β	0.17	0.42	1.62	0.6
N'_p	− 0.17	− 0.07	− 0.016	− 0.032
N'_r	− 0.217	− 0.14	− 0.232	− 0.115
N'_{δ_A}	0.026	0.018	0.013	0.008
N'_{δ_R}	− 0.15	− 0.39	− 0.92	− 0.48

B.2.4 DELTA – A very large, four-engined, cargo jet aircraft

General Parameters

Wing area (m^2)	576	
Aspect ratio:	7.75	
Chord, $\bar{c}$ (m):	9.17	
Total related thrust (kN):	730	
C.g.:	$0.3\bar{c}$	
Pilot's location (m) (relative to c.g.)		
l_{x_P}:	25.0	
l_{z_P}:	+ 2.5	
Weight (kg):	*Approach* 264 000	*All other flight conditions* 300 000

Inertias (kg m^2)

I_{xx}:	2.6×10^7	3.77×10^7
I_{yy}:	4.25×10^7	4.31×10^7
I_{zz}:	6.37×10^7	7.62×10^7
I_{xz}:	3.4×10^6	3.35×10^6

Flight Conditions

Parameter	*Flight condition*			
	1	*2*	*3*	*4*
Height (m)	S.L.	6 100	6 100	12 200
Mach no.	0.22	0.6	0.8	0.875
U_0 (m s^{-1})	75	190	253	260
$\bar{q}$ (N m^{-2})	3 460	11 730	20 900	10 100
α_0 (degrees)	+ 2.7	+ 2.2	+ 0.1	+ 4.9
γ_0 (degrees)	0	0	0	0

Stability Derivatives

Longitudinal Motion

Stability derivative	*Flight condition*			
	1	*2*	*3*	*4*
X_u	− 0.02	− 0.003	− 0.02	− 0.03
X_w	0.1	0.04	0.02	0.0
X_{δ_E}	0.14	0.26	0.32	0.45
$X_{\delta_{th}}$	0.17×10^{-4}	0.15×10^{-4}	0.15×10^{-4}	0.15×10^{-4}
Z_u	− 0.23	− 0.08	− 0.01	0.17
Z_w	− 0.634	− 0.618	− 0.925	− 0.387
Z_{δ_E}	− 2.9	− 6.83	− 9.51	− 5.18
$Z_{\delta_{th}}$	0.06×10^{-5}	0.05×10^{-5}	0.05×10^{-5}	0.05×10^{-5}
M_u	-2.55×10^{-5}	3.28×10^{-4}	14.21×10^{-4}	54.79×10^{-4}
M_w	− 0.005	− 0.007	− 0.0011	− 0.006
$M_{\dot{w}}$	− 0.003	− 0.001	− 0.001	− 0.0005
M_q	− 0.61	− 0.77	− 1.02	− 0.55
M_{δ_E}	− 0.64	− 1.25	− 1.51	− 0.92
$M_{\delta_{th}}$	1.44×10^{-5}	1.42×10^{-5}	1.42×10^{-5}	1.42×10^{-5}

Lateral Motion

Stability derivative	*Flight condition* *1*	*2*	*3*	*4*
Y_v	− 0.078	− 0.11	− 0.15	− 0.07
$Y^*_{\delta_A}$	− 0.0001	-0.29×10^{-4}	-0.38×10^{-4}	-0.18×10^{-4}
$Y^*_{\delta_R}$	0.0065	0.0055	0.006	0.002
L'_β	− 0.635	− 1.33	− 2.38	0.333
L'_p	− 1.09	− 1.0	− 1.42	− 0.63
L'_r	0.613	0.28	0.30	0.26
L'_{δ_A}	0.46	0.43	0.37	0.36
L'_{δ_R}	0.1	0.187	0.29	0.107
N'_β	0.11	0.432	0.885	0.386
N'_p	− 0.16	− 0.09	− 0.09	− 0.07
N'_r	− 0.23	− 0.2	− 0.25	− 0.009
N'_{δ_A}	0.05	0.03	0.09	0.04
N'_{δ_R}	− 0.21	− 0.52	− 0.83	− 0.34

B.2.5 ECHO – A single-engined, CCV, jet fighter aircraft

General Parameters

Wing area (m^2):	26
Aspect ratio:	3.0
Chord, $\bar{c}$ (m):	3.33
Total related thrust (kN):	11
C.g.:	$0.35\,\bar{c}$
Pilot's location (m) (relative to c.g.)	
l_{x_p}:	3.9
l_{z_p}:	− 0.326
Weight (kg):	84.52
Inertias (kg m^2):	
I_{xx}:	11×10^3
I_{yy}:	6.38×10^4
I_{zz}:	7.24×10^4
I_{xz}:	4.7×10^4

Flight Conditions

Parameter	*Flight condition* 1	2	3	4
Height (m)	S.L.	4 600	9 100	15 250
Mach no.	0.6	0.8	0.95	1.7
U_0 (m s^{-1})	207	258	288	502
$\bar{q}$ (N m^{-2})	26 245	25 860	17 362	23 400
α_0 (degrees)	+ 1.92	+ 2.17	+ 4.25	+ 1.6
γ_0 (degrees)	0	0	0	0

Stability Derivatives

Longitudinal Motion only

Stability derivative	*Flight condition* 1	2	3	4
Z_α	− 0.0272	− 0.023	− 0.016	− 0.008
$Z_{\dot{\alpha}}$	− 0.484	− 0.295	− 0.288	0.19
Z_q	− 2.605	− 1.866	− 1.5	− 0.46
Z_{δ_E}	− 0.721	− 0.67	− 0.4	− 0.4
Z_{δ_F}	− 0.925	− 0.95	− 0.612	0.0
M_α	0.0055	0.0005	− 0.0002	− 0.0018
$M_{\dot{\alpha}}$	− 0.136	− 0.348	− 0.318	0.726
M_q	− 1.013	− 0.952	− 0.913	− 1.014
M_{δ_E}	− 0.364	− 0.362	− 0.251	− 0.66
M_{δ_F}	− 0.034	− 0.056	− 0.084	0.0

B.2.6 FOXTROT – A twin-engined, jet fighter/bomber aircraft

General Parameters

Wing area (m^2):	49.24
Aspect ratio:	4.0
Chord, $\bar{c}$ (m):	4.88
Total related thrust (kN):	160
C.g.:	0.29 $\bar{c}$
Pilot's location (m) (relative to c.g.)	
l_{x_P}:	5.32
l_{z_P}:	− 1.0

	Approach	*All other flight conditions*
Weight (kg):	148	173
Inertias (kg m^2):		
I_{xx}:	32 100	33 900
I_{yy}:	16 000	166 000
I_{zz}:	181 400	190 000
I_{xz}:	2 100	3 000

Flight Conditions

Parameter	*Flight condition*			
	1	*2*	*3*	*4*
Height (m)	S.L.	10 650	10 650	13 700
Mach no.	0.206	0.9	1.2	2.15
U_0 (m s^{-1})	70	265	350	650
$\bar{q}$ (N m^{-2})	2 997	13 550	24 090	48 070
α_0 (degrees)	11.7	2.6	1.6	1.4
γ_0 (degrees)	0	0	0	0

Stability Derivatives

Longitudinal Motion

Stability derivative	*Flight condition*			
	1	*2*	*3*	*4*
X_u	− 0.042	− 0.009	− 0.0135	0.016
X_w	0.14	0.016	0.006	0.004
Z_u	− 0.177	− 0.088	0.0125	− 0.001
Z_w	− 0.452	− 0.547	− 0.727	− 0.494
Z_q	− 0.76	− 0.88	− 1.25	− 0.39
M_u	0.0024	− 0.008	0.009	0.07
M_w	− 0.006	− 0.03	− 0.08	− 0.07
$M_{\dot{w}}$	− 0.002	− 0.001	− 0.001	0.001
M_q	− 0.317	− 0.487	− 0.745	− 0.41
$X_{\delta_{th}}$	0.00007	0.00006	0.00006	0.00006
$Z_{\delta_{th}}$	− 0.0006	− 0.00005	− 0.00005	− 0.00005
$M_{\delta_{th}}$	− 0.00005	− 0.000003	− 0.000003	− 0.000003
X_{δ_E}	1.83	0.69	0.77	0.62
Z_{δ_E}	− 2.03	− 15.12	− 27.55	− 25.45
M_{δ_E}	− 1.46	− 11.4	− 20.7	− 16.1

Lateral Motion

Stability derivative	Flight condition 1	2	3	4
Y_β	− 21.1	− 80.6	− 176.0	− 277.0
L'_β	− 10.4	− 18.3	− 14.1	− 8.67
L'_p	− 1.43	− 1.24	− 1.38	− 1.08
L'_r	0.929	0.395	0.318	0.22
N'_β	1.44	4.97	12.3	8.37
N'_p	− 0.026	− 0.0504	− 0.038	0.015
N'_r	− 0.215	− 0.238	− 0.4	− 0.275
$Y^*_{\delta_A}$	− 0.004	− 0.0007	− 0.0009	− 0.0005
$Y^*_{\delta_R}$	0.0053	0.0043	0.004	0.0026
L'_{δ_A}	2.74	9.0	10.9	5.35
L'_{δ_R}	0.7	1.95	3.0	2.6
N'_{δ_A}	0.42	0.2	0.67	0.36
N'_{δ_R}	− 0.67	− 2.6	− 3.2	− 1.86

B.2.7 GOLF – A twin-piston engined, general aviation aircraft

General Parameters

Wing area (m^2)	21.0	
Aspect ratio:	8.2	
Chord, $\bar{c}$ (m):	1.77	
Total related thrust (kN):	48.5	
C.g.:	$0.25\bar{c}$	
Pilot's location (m) (relative to c.g.)		
l_{x_p}:	1.0	
l_{z_p}:	− 0.3	
Weight (kg):	*Approach* 2000	*All other flight conditions* 2775
Inertias (kg m^2)		
I_{xx}:	13 470	20 420
I_{yy}:	20 450	27 560
I_{zz}:	27 200	46 000
I_{xz}:	2 150	5 870

Flight Conditions

Parameter	*Flight condition* 1	2	3	4
Height (m)	S.L.	S.L.	1 600	6 500
Mach no.	0.143	0.19	0.207	0.345
U_0 (m s^{-1})	50.0	65	70	105
$\bar{q}$ (N m^{-2})	1 530	2 590	1 960	3 440
α_0 (degrees)	—	—	—	—
γ_0 (degrees)	—	—	—	—

Stability Derivatives

Longitudinal Motion

Stability derivative	*Flight condition* 1	2	3	4
X_u	− 0.053	− 0.023	− 0.021	− 0.018
X_α	21.01	12.8	12.57	18.34
Z_u	− 0.002	− 0.001	− 0.001	− 0.005
Z_w	− 1.05	− 1.333	− 1.241	− 1.234
Z_q	− 0.024	− 0.025	− 0.021	− 0.012
M_u	0.016	0.0076	0.005	0.003
M_α	− 12.3	− 21.26	− 23.46	− 38.43
M_q	− 6.22	− 8.15	− 7.58	− 7.2
X_{δ_E}	− 0.046	− 0.061	− 0.055	− 0.052
X_{δ_F}	− 0.017	− 0.08	− 0.074	− 0.074
Z_{δ_F}	− 0.96	− 1.811	− 1.811	− 2.83
Z_{δ_E}	− 1.04	− 2.24	− 2.2	− 3.1
M_{δ_E}	− 13.55	− 23.4	− 23.5	− 34.85
M_{δ_F}	1.0	1.414	1.29	1.55

Lateral Motion

Stability derivative	*Flight condition* 1	2	3	4
Y_v	− 0.145	− 0.188	− 0.174	− 0.184
Y_p	0.087	0.087	0.09	0.05
L'_β	− 2.18	− 3.71	− 3.71	− 5.33
L'_p	− 2.01	− 2.63	− 2.43	− 2.33
L'_r	0.303	0.39	0.36	0.31
N'_β	2.182	3.71	3.71	6.33
N'_p	− 0.222	− 0.29	− 0.27	− 0.17
N'_r	− 0.27	− 0.35	− 0.325	− 0.314
$Y^*_{\delta_R}$	0.038	0.049	0.049	0.045
L'_{δ_A}	1.541	2.63	2.62	4.16
L'_{δ_R}	0.6	1.02	1.02	1.6
N'_{δ_A}	− 0.036	− 0.036	− 0.061	− 0.044
N'_{δ_R}	− 1.25	− 1.25	− 2.1	− 3.33

Appendix C

Mathematical Models of Human Pilots

C.1 INTRODUCTION

Notwithstanding the extent to which flight control is being made automatic, it remains essential for the designers of flight control systems to remember that a human pilot acts as the 'outer loop' of a complete flight control system. As AFCSs have been improved and developed, the need to represent human pilots by appropriate mathematical models has become more pressing, although the need for such representation has been recognized for a considerable time. It has been the cause of a great amount of research which is recorded in a most extensive literature. Chief among the workers researching in this field have been McRuer, Krendel and Graham, and it is their work (see the various references at the end of this appendix) which provides the basis for those models dealt with briefly below. More extensive models exist, such as Paper Pilot (Dillow, 1971), but they are beyond the scope of an introductory textbook such as this.

There are several reasons for using a mathematical model in studies relating to the performance of closed loop flight control systems being operated by a human pilot; the include the following:

1. The prediction of what may be possible from some given arrangement.
2. The evolution and, perhaps, development of critical flight or simulator experiments.
3. The interpretation of flight tests or simulator results.
4. The determination of the limitations of validity of any experimental results.

From examining the nature of a pilot's behaviour when flying it becomes clear that he normally demonstrates those characteristics commonly described as adaptive and multimodal. Even when carrying out familiar tasks, the pilot is also capable of learning. This knowledge suggests that the construction of any appropriate mathematical model may incorporate some of the following features:

1. The differential equations involved should be invariant , or time-varying.
2. The model may be multi- or single-variable.
3. The equations may be linear or non-linear.
4. The data may be continuous or sampled.

The model should represent adequately the pilot's actions when carrying out a pursuit task or controlling the aircraft using a compensatory display. From extensive experiments on human operators it has been learned that one appropriate form of model was a describing function which represents the linear response of the operator whose actual response can only be accurately described by non-linear equations. But these describing functions represent very good approximations for most pilot actions. The validity of the describing function model does depend upon the addition of a remnant term, but, for simplicity, only the linear models represented by describing functions are used here. A remnant term can be considered to be a bias term to ensure that the describing function corresponds to the appropriate operating point. One example of how such a term can be included in the model is given in paragraph 4 below.

C.2 CLASSICAL MODELS

1. The pilot's response is denoted by v_p; his command is taken as p_{comm}. Basically, the model assumes that the response is linear and proportional to the command, with some prediction, but with a pure time delay caused by the finite reaction time of the pilot. The model is represented in Figure C.1 from which it can be deduced that

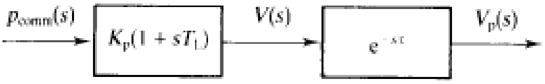


Figure C.1 Block diagram of pilot model – lead term and pure time delay.

$$\frac{V_p(s)}{p_{comm}(s)} = K_p(1 + sT_L)e^{-s\tau} \tag{C.1}$$

The transfer function representing the pure time delay, namely:

$$V_p(s)/V(s) = e^{-s\tau} \tag{C.2}$$

is a transcendental function and can only be completely represented by an infinite series. Consequently, a suitable approximation is needed. One of the most accepted is the first order Padé approximation:

$$V_p(s)/V(s) = -(s - 2/\tau)/(s + 2/\tau) \tag{C.3}$$

2. Refer to Figure C.2.

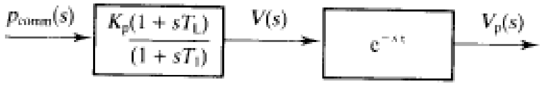


Figure C.2 Block diagram of pilot model – phase advance and pure time delay.

$$\frac{V_p(s)}{p_{comm}(s)} = K_p \frac{(1 + sT_L)}{(1 + sT_1)} e^{-s\tau} = \frac{V(s)}{p_{comm}(s)} \cdot \frac{V_p(s)}{V(s)} \tag{C.10}$$

3. Refer to Figure C.3.

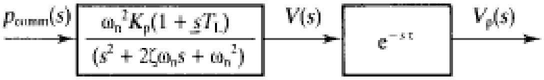


Figure C.3 Block diagram of pilot model—lead term, pure time delay and neuromuscular lag.

$$\frac{V_p(s)}{p_{comm}(s)} = \frac{\omega_n^2 K_p(1 + sT_L)e^{-s\tau}}{(s^2 + 2\zeta s\omega_n + \omega_n^2)} = \frac{V_p(s)}{V(s)} \frac{V(s)}{p_{comm}(s)} \tag{C.16}$$

The term:

$\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$

represents the addition of a neuromuscular lag to the model.

C.3 REFERENCES

DILLOW, J.D. 1971. 'Super Pilot' – a revised version of Paper Pilot. AFFDL/FGC-TM-71-9, WPAFB, Dayton, Ohio.

KRENDEL, E.S. and D.T. McRUER. 1960. A servomechanisms approach to skills development. *J. Frank. Inst.* 269 (1): 24–42.

McRUER, D.T. and D. GRAHAM. 1964. Pilot vehicle control system analysis. *Progress in Astronautics and Aeronautics*, Vol. 13, Guidance and Control II. New York: Academic Press, pp. 603–21.

McRUER, D.T., D. GRAHAM and E.S. KRENDEL. 1967. Manual control of single loop systems. *J. Frank. Inst.* 283 (182): 1–29, 145–68.

McRUER, D.T. and E.S. KRENDEL. 1974. Mathematical models of human pilot behaviour. *Agardograph* no. 188.

McRUER, D.T., R.E. MAGDALENO and G.P. MOORE. 1968. A neuro-muscular actuation system model. *Trans IEEE.* MMS-9(3): 61–71.